Proofs, and what they're good for

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MELBOURNE PHILOSOPHY SEMINAR · AUGUST 2016
To explain the nature of proof, from the perspective of a normative pragmatic account of meaning, using the formal tools of proof theory.
MOTIVATION
Every drink (in our fridge) is either a beer or a lemonade.
Example Proof 1

Every *drink* (in our fridge) is either a *beer* or a *lemonade*

\[(\forall x)(Dx \supset (Bx \lor Lx))\].

Why?
Take an arbitrary drink. It's either a beer or a lemonade. If it's a lemonade, we have the conclusion that some drink is a lemonade. If we don't have that conclusion, then that arbitrary drink is a beer, and so, all the drinks are beers, and so, we also have our conclusion.
Every *drink* (in our fridge) is either a *beer* or a *lemonade* 
\( \forall x (Dx \supset (Bx \lor Lx)) \). So either every *drink* is a *beer* or some *drink* is a *lemonade*
Every *drink* (in our fridge) is either a *beer* or a *lemonade* $(\forall x)(Dx \supset (Bx \lor Lx))$. So either every *drink* is a *beer* or some *drink* is a *lemonade* $(\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)$. 
Every drink (in our fridge) is either a beer or a lemonade $(\forall x)(Dx \supset (Bx \lor Lx))$. So either every drink is a beer or some drink is a lemonade $(\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)$. Why?
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\((\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)\). Why? Take an arbitrary *drink*. It’s either a *beer* or a *lemonade*. If it’s a *lemonade*, we have the conclusion that some *drink* is a *lemonade*. If we don’t have that conclusion, then that arbitrary *drink* is a *beer*, and so, all the *drinks* are *beers*, and so, we also have our conclusion.
Example Proof 1 (the formal structure)

\[
\begin{align*}
&\quad \text{Ba} \rightarrow \text{Ba} \quad \text{La} \rightarrow \text{La} \\
&Da \rightarrow Da \quad \text{Ba} \lor \text{La} \rightarrow \text{Ba}, \text{La} \\
&\quad \therefore \text{Da} \supset (\text{Ba} \lor \text{La}), \text{Da} \rightarrow \text{Ba}, \text{La} \\
&(\forall x) (\text{Dx} \supset (\text{Bx} \lor \text{Lx})), \text{Da} \rightarrow \text{Ba}, \text{La} \\
&Da \rightarrow Da \\
&\quad \therefore (\forall x) (\text{Dx} \supset (\text{Bx} \lor \text{Lx})), \text{Da} \rightarrow \text{Ba}, \text{Da} \land \text{La} \\
&(\forall x) (\text{Dx} \supset (\text{Bx} \lor \text{Lx})), \text{Da} \rightarrow \text{Ba}, (\exists x) (\text{Dx} \land \text{Lx}) \\
&\quad \therefore (\forall x) (\text{Dx} \supset (\text{Bx} \lor \text{Lx})), \text{Da} \supset \text{Ba}, (\exists x) (\text{Dx} \land \text{Lx}) \\
&(\forall x) (\text{Dx} \supset (\text{Bx} \lor \text{Lx})), (\exists x) (\text{Dx} \land \text{Lx}) \\
&\quad \therefore (\forall x) (\text{Dx} \supset \text{Bx}) \lor (\exists x) (\text{Dx} \land \text{Lx})
\end{align*}
\]
Example Proof 2 (The Bridges of Königsberg)

It’s not possible to walk a circuit through Königsberg, crossing each bridge exactly once.
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It’s not possible to walk a circuit through Königsberg, *crossing each bridge exactly once*. Why? Any bridge takes you from one landmass (A, B, C, D) to another. In any circuit, you must *leave* a landmass as many times as you *arrive*. 
It’s not possible to walk a circuit through Königsberg, *crossing each bridge exactly once*. Why? Any bridge takes you from one landmass (A, B, C, D) to another. In any circuit, you must *leave* a landmass as many times as you *arrive*. So, if you are use every bridge exactly once, each landmass must have an even number of bridges entering and exiting it.
Example Proof 2 (The Bridges of Königsberg)

It’s not possible to walk a circuit through Königsberg, *crossing each bridge exactly once.* Why? Any bridge takes you from one landmass (\(A, B, C, D\)) to another. In any *circuit,* you must *leave* a landmass as many times as you *arrive.* So, if you are use every bridge exactly once, each landmass must have an even number of bridges entering and exiting it. Here, each landmass has an *odd* number of bridges, so a circuit is impossible.
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But what I say here can be extended to proof relying on other concepts.
Puzzles about proof

- How can proofs expand our knowledge, when the conclusion is (in some sense) already present in the premises?
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- What grounds the necessity in the connection between the premises and the conclusion?
Puzzles about proof

▶ How can proofs expand our knowledge, when the conclusion is (in some sense) already present in the premises?

▶ How can we be ignorant of a conclusion which logically follows from what we already know?

▶ What grounds the necessity in the connection between the premises and the conclusion?

▶ (Notice that these are important questions for proofs in first order predicate logic, as much as for proof more generally.)
BACKGROUND
Positions ...

Assertions and Denials

\[X : Y\]
Assertions and denials are moves in a practice.
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I can deny what you assert.

They are connected to other speech acts, like imperatives, interrogatives, recognitives, observatives, etc.
Assertions and denials are moves in a practice.

I can *deny* what you *assert*.

I can *retract* an assertion or a denial.
Assertions and denials are moves in a practice.

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I can ‘try on’ assertion or denial hypothetically.
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I can *deny* what you *assert*.

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I can ‘try on’ assertion or denial hypothetically.

They are connected to other speech acts, too, like imperatives, interrogatives, recognitives, observatives, *etc.*
Assertions and denials take a *stand* *(pro or con)* on something.

DENIAL clashes with assertion.

ASSERTION clashes with denial.
These norms give rise to BOUNDS for positions.
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**Identity:** $[A : A]$ is out of bounds.
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**WEAKENING:** If $[X : Y]$ is out of bounds, then $[X, A : Y]$ and $[X : A, Y]$ are also out of bounds.
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**Weakening:** If $[X : Y]$ is out of bounds, then $[X, A : Y]$ and $[X : A, Y]$ are also out of bounds.

**Cut:** If $[X, A : Y]$ and $[X : A, Y]$ are out of bounds, then so is $[X : Y]$. 
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**CUT:** If $[X, A : Y]$ and $[X : A, Y]$ are out of bounds, then so is $[X : Y]$.

A position that is OUT OF BOUNDS doesn’t succeed in taking a stand.
Explicit Definition

Define a concept by showing how you can compose that concept out of more primitive concepts.
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\[ x \text{ is a square} \equiv_{df} x \text{ is a rectangle} \land \text{all sides of } x \text{ are equal in length}. \]
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Concepts defined explicitly are *sharply delimited* (contingent on the definition).
Explicit Definition

Define a concept by showing how you can compose that concept out of more primitive concepts.

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all sides of \( x \) are equal in length.

Concepts defined explicitly are *sharply delimited*
(contingent on the definition).

Logical concepts are similarly sharply delimited, but they cannot all be given explicit definitions.
Definition through a rule for use

\[ [X, A \land B : Y] \text{ is out of bounds if and only if } [X, A, B : Y] \text{ is out of bounds} \]
Definition through a rule for use

\[(X, A \land B : Y)\] is out of bounds if and only if

\[(X, A, B : Y)\] is out of bounds

\[
\frac{X, A, B \vdash Y}{X, A \land B \vdash Y} \land \text{Df}
\]
What about when to deny a conjunction?

\[
\begin{align*}
&\quad X \vdash A; Y \\
A \land B &\vdash A \land B &\text{Id} \\
X \vdash B, Y &\quad A, B \vdash A \land B &\text{\&Df} \\
X \vdash A, Y &\quad X, A \vdash A \land B, Y &\text{Cut} \\
&\quad \quad X \vdash A \land B, Y &\text{Cut}
\end{align*}
\]
What about when to *deny* a conjunction?

So, we have

$$
\begin{align*}
X \vdash A, Y & \quad X \vdash B, Y \\
\hline
& X \vdash A \land B, Y
\end{align*}
$$
Definitions for other logical concepts

\[ \begin{array}{c}
X \vdash A, Y \\
\Rightarrow Df
\end{array} \]

\[ \begin{array}{c}
X, \neg A \vdash Y \\
\Rightarrow Df
\end{array} \]

\[ \begin{array}{c}
X, A \vdash B, Y \\
\Rightarrow Df
\end{array} \]

\[ \begin{array}{c}
X \vdash A \supset B, Y \\
\Rightarrow Df
\end{array} \]

\[ \begin{array}{c}
X \vdash A \lor B, Y \\
\lor Df
\end{array} \]

(Where \( a \) and \( G \) are not present in \( X \) and \( Y \).)
(Where $a$ and $G$ are not present in $X$ and $Y$.)
Concepts defined in this way...

- Are *uniquely defined*. (If you and I use the same rule, we define the same concept.)

- Are conservatively extending. (Adding a logical concept to your vocabulary in this way doesn’t constrain the bounds in the original language.)

- Play useful dialogical roles. (You can do things with these concepts that you cannot do without. Denying a conjunction does something different to denying the conjuncts.)

- Are subject matter neutral. (They work wherever you assert and deny—and have singular terms and predicates.)

- In Brandom’s terms, they make explicit some of what was implicit in the practice of assertion and denial.
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- In Brandom’s terms, they *make explicit* some of what was implicit in the practice of assertion and denial.
WHAT PROOFS ARE
If it’s Thursday, I’m in Melbourne.

It’s Thursday.

Therefore, I’m in Melbourne.
A Tiny Proof

If it’s Thursday, I’m in Melbourne.

It’s Thursday.

Therefore, I’m in Melbourne.

\[
\begin{align*}
A \supset B & \vdash A \supset B \\
\frac{}{A \supset B, A \vdash B} & \text{Id} \\
\frac{}{A \supset B} & \text{\supset Def}
\end{align*}
\]
If it’s Thursday, I’m in Melbourne.

It’s Thursday.

Therefore, I’m in Melbourne.

\[
\begin{align*}
A \supset B & \vdash A \supset B & \text{Id} \\
A \supset B, A \vdash B & \vdash \text{Df}
\end{align*}
\]

[It’s Thursday \supset I’m in Melbourne, It’s Thursday : I’m in Melbourne]

(This is out of bounds.)
Take a context in which I’ve asserted

\( \textit{it’s Thursday} \supset \textit{I’m in Melbourne} \)

and I’ve asserted \( \textit{it’s Thursday} \),
then \( \textit{I’m in Melbourne} \) is undeniable.
Take a context in which I’ve asserted

\[ \text{it’s Thursday} \supset \text{I’m in Melbourne} \]

and I’ve asserted \text{it’s Thursday},
then \text{I’m in Melbourne} is undeniable.

Adding the \textit{assertion} makes explicit what was \textit{implicit} before that assertion.
Take a context in which I’ve asserted it’s Thursday ⊃ I’m in Melbourne and I’ve asserted it’s Thursday, then I’m in Melbourne is undeniable.

Adding the assertion makes explicit what was implicit before that assertion.

The stance (pro or con) on I’m in Melbourne was already made.
A proof of $X \vdash Y$ shows that the position $[X : Y]$ is out of bounds, by way of the defining rules for the concepts involved in the proof.
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In this sense, proofs are *analytic*. They apply, given the definitions, independently of the positions taken by those giving the proof.
A proof of $A, B \vdash C, D$ can be seen as a proof of $C$ from $[A, B : D]$, and a refutation of $A$ from $[B : C, D]$, and more.
HOW PROOFS WORK
Observation 0: Proofs are *analytic*

These proofs are grounded in the *rules defining* the concepts used in them.
Observation 1: *Specification* outstrips *Recognition*

Our ability to *specify* concepts and consequence far outstrips our ability to *recognise* that consequence.
Peano Arithmetic and Goldbach's Conjecture

**SUCCESSOR AXIOMS:**

**PA1:** \( \forall x \forall y (x' = y' \supset x = y) \);

**PA2:** \( \forall x (0 \neq x') \).

**ADDITION AXIOMS:**

**PA3:** \( \forall x (x + 0 = x) \);

**PA4:** \( \forall x (x + y' = (x + y)') \).

**MULTIPLICATION AXIOMS:**

**PA5:** \( \forall x (x \times 0 = 0) \);

**PA6:** \( \forall x \forall y (x \times y' = (x \times y) + x) \).

**INDUCTION SCHEME:**

**PA7:** \( (\phi(0) \land \forall x (\phi(x) \supset \phi(x'))) \supset \forall x \phi(x) \).

**GOLDBACH’S CONJECTURE:**

**GC:** \( \forall x \exists y \exists z (\text{Prime } y \land \text{Prime } z \land 0'' \times x = y + z) \)
Observation 1: *Specification outstrips Recognition*

Is \([PA : GC]\) out of bounds?
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Is \([PA : GC]\) out of bounds?

We have no idea.
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We have no idea.

This is not a *bug*. It’s a *feature*. 
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We have no idea.

This is not a *bug*. It’s a *feature*.

Our concepts are rich and expressive. We can say things whose significance we continue to work out.
Observation 1: *Specification* outstrips *Recognition*

Is \([\text{PA} : \text{GC}]\) out of bounds?

We have no idea.

This is not a *bug*. It’s a *feature*.

Our concepts are rich and expressive.
We can say things whose significance we continue to work out.

Verifying a putative proof is straightforward.
Checking that something *has* a proof is not so easy.
Are we logically omniscient?

Suppose that $\text{PA} \vdash \text{GC}$
(but we don’t possess that proof)
and that we know $\text{PA}$.

Do we know $\text{GC}$?
In a weak sense of ‘know’, yes, we do know Gc.

- It’s a logical consequence of what we know.
In a weak sense of ‘know’, yes, we do know \( \text{GC} \)

- It’s a logical consequence of what we know.
- It is implicitly present in what we already know.
In a weak sense of ‘know’, yes, we do know GC

- It’s a logical consequence of what we know.
- It is implicitly present in what we already know.
- There is no epistemic possibility (no circumstance consistent with our knowledge) that leaves GC out.
In a not-so-weak sense, we don't know GC

- Do we believe GC?
In a not-so-weak sense, we don't know GC

- Do we believe GC?
- If we believed it, do we believe it *in the right way*?
In a not-so-weak sense, we don't know GC

- Do we believe GC?
- If we believed it, do we believe it *in the right way*?
- There is evidence for GC (its proof from PA, for example), but if that evidence plays no role in our belief...
Observation 2: Proofs Preserve Truth

- The account of consequence does not use the concept of \textit{truth}.
Observation 2: Proofs Preserve Truth

- The account of consequence does not use the concept of *truth*.

- However, given plausible (minimal) assumptions concerning $T$, we can show that *(for example)* if $A, B \vdash C$ then $T(A), T(B) \vdash T(C)$. 

This follows from the concepts of consequence and truth.
Observation 2: Proofs Preserve Truth

- The account of consequence does not use the concept of truth.

- However, given plausible (minimal) assumptions concerning T, we can show that (for example) if $A, B \vdash C$ then $T\langle A \rangle, T\langle B \rangle \vdash T\langle C \rangle$.

- This follows from the concepts of consequence and truth.
Observation 3: Proofs Can Preserve Warrant

- The account of consequence does not use the concept of warrant.
Observation 3: Proofs Can Preserve Warrant

- The account of consequence does not use the concept of *warrant*.

- However, given plausible (less minimal) assumptions concerning warrant, we can show that *(for example)* if \( p \) is a proof for \( A, B \vdash C \) then \( x : A, y : B \vdash p(x, y) : C \).
Observation 3: Proofs Can Preserve Warrant

- The account of consequence does not use the concept of warrant.

- However, given plausible (less minimal) assumptions concerning warrant, we can show that (for example) if \( p \) is a proof for \( A, B \vdash C \) then \( x : A, y : B \vdash p(x, y) : C \).

- Here, \( p \) transforms warrants for the premises into warrant for the conclusion.
Observation 3: Proofs Can Preserve Warrant

- The account of consequence does not use the concept of warrant.

- However, given plausible (less minimal) assumptions concerning warrant, we can show that (for example) if $p$ is a proof for $A, B \vdash C$ then $x : A, y : B \vdash p(x, y) : C$.

- Here, $p$ transforms warrants for the premises into warrant for the conclusion.

- This works only for categorical, conclusive warrants (grounds), not for defeasible warrants.
Consider the “Lottery Paradox.”
A Caveat on Defeasible Warrants

Consider the “Lottery Paradox.”

\[
\begin{align*}
(\exists x) (T_x \land W_x), \\
(\forall x) (T_x \equiv (x = t_1 \lor x = t_2 \lor \cdots \lor x = t_{1,000,000})) \\
: W_{t_1}, W_{t_2}, \ldots, W_{t_{1,000,000}}
\end{align*}
\]
Consider the “Lottery Paradox.”

\[
(\exists x)(Tx \land Wx),
(\forall x)(Tx \equiv (x = t_1 \lor x = t_2 \lor \cdots \lor x = t_{1,000,000}))
\]

\[
: Wt_1, Wt_2, \ldots, Wt_{1,000,000}
\]

We have a very high degree of confidence in each part. Each component is highly probable. But the whole position is out of bounds.
“Well, now, let’s take a little bit of the argument in that First Proposition—just two steps, and the conclusion drawn from them. Kindly enter them in your note-book. And in order to refer to them conveniently, let’s call them $A$, $B$, and $Z$:

$(A)$ Things that are equal to the same are equal to each other.

$(B)$ The two sides of this Triangle are things that are equal to the same.

$(Z)$ The two sides of this Triangle are equal to each other.

Readers of Euclid will grant, I suppose, that $Z$ follows logically from $A$ and $B$, so that any one who accepts $A$ and $B$ as true, must accept $Z$ as true?”

“Undoubtedly! The youngest child in a High School—as soon as High Schools are invented, which will not be till some two thousand years later—will grant that.”

“And if some reader had not yet accepted $A$ and $B$ as true, he might still accept the sequence as a valid one, I suppose?”
“No doubt such a reader might exist. He might say ‘I accept as true the Hypothetical Proposition that, if $A$ and $B$ be true, $Z$ must be true; but, I \textit{don’t} accept $A$ and $B$ as true.’ Such a reader would do wisely in abandoning Euclid, and taking to football.”

“And might there not \textit{also} be some reader who would say ‘I accept $A$ and $B$ as true, but I \textit{don’t} accept the Hypothetical’?”

“Certainly there might. \textit{He}, also, had better take to football.”

“And \textit{neither} of these readers,” the Tortoise continued, “is \textit{as yet} under any logical necessity to accept $Z$ as true?”

“Quite so,” Achilles assented.

“Well, now, I want you to consider \textit{me} as a reader of the \textit{second} kind, and to force me, logically, to accept $Z$ as true.”

“A tortoise playing football would be—” Achilles was beginning

“—an anomaly, of course,” the Tortoise hastily interrupted. “Don’t wander from the point. Let’s have $Z$ first, and football afterwards!”

“I’m to force you to accept $Z$, am I?” Achilles said musingly. “And your present position is that you accept $A$ and $B$, but you \textit{don’t} accept the Hypothetical—”

“Let’s call it $C$,“ said the Tortoise.

“—but you \textit{don’t} accept

$(C)$ \textit{If $A$ and $B$ are true, $Z$ must be true.}”

“That is my present position,” said the Tortoise.

“Then I must ask you to accept $C$.”
Our Analysis

\[ A, B \vdash Z \]

This doesn't mean when I accept \( A \) and I accept \( A \), I ought to also accept \( Z \). However, if I assert \( A \) and \( A \) then \( Z \) is undeniable.
Our Analysis

\[ A, B \vdash Z \]

or

\[ A, A \supset Z \vdash Z \]
Our Analysis

$A, B \vdash Z$

or

$A, A \supset Z \vdash Z$

This *doesn’t* mean when I accept $A$ and I accept $A \supset Z$, I ought to also accept $Z$. 
Our Analysis

\[ \begin{align*}
A, B & \vdash Z \\
\text{or} \\
A, A \supset Z & \vdash Z
\end{align*} \]

This *doesn’t* mean when I accept \(A\) and I accept \(A \supset Z\), I ought to also accept \(Z\).

However, if I assert \(A\) and \(A \supset Z\) then \(Z\) is *undeniable*. 
If I assert $A$ and $if A \text{ then } Z$ and deny $Z$, then I am using ‘if ... then’ in a way that deviates from the defining rule for $⊃$, or I am explicitly contradicting myself.
If I assert $A$ and *if* $A$ *then* $Z$ and deny $Z$, then I am using ‘*if* ... *then*’ in a way that deviates from the defining rule for $\supset$, or I am explicitly contradicting myself.

$$
A \supset B \vdash A \supset B
\quad \text{\textit{Df}}
$$

$$
A \supset B, A \vdash B
\quad \supset Df
$$
An account of the logical concepts given in terms of defining rules governing assertions and denials helps explain how (first order predicate logic) proof works, how possessing a proof can expand our knowledge, while proofs make explicit what is implicit in what we know.
Upshot

An account of the logical concepts given in terms of defining rules governing assertions and denials helps explain how (first order predicate logic) proof works,
An account of the logical concepts given in terms of defining rules governing assertions and denials helps explain how (first order predicate logic) proof works, how possessing a proof can expand our knowledge, while proofs make explicit what is implicit in what we know.
THANK YOU!


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