# Proofs for Relevant Consequence with Star and Perp

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University of
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Logic, Reasoning, and Justification

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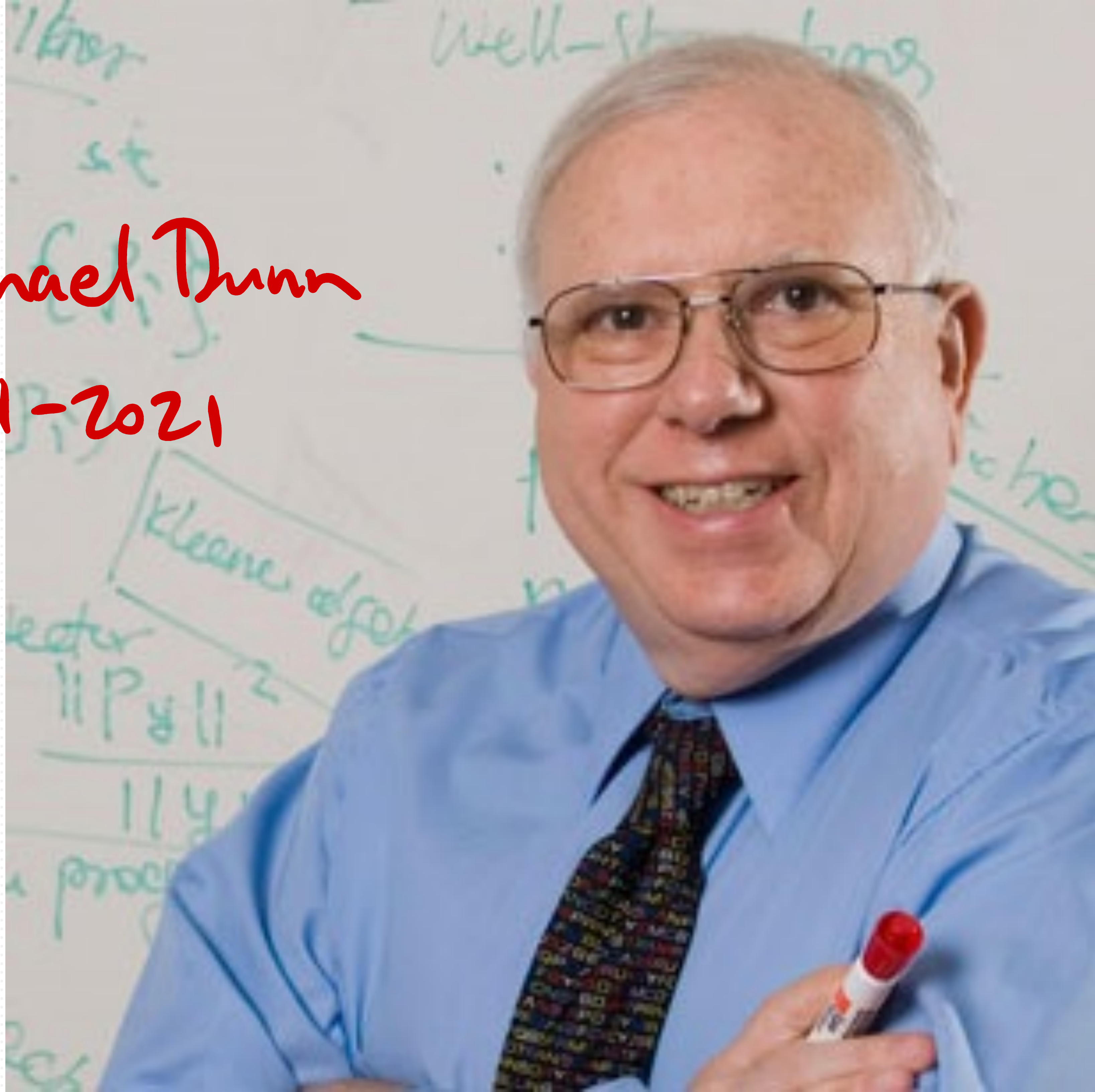
# My Goal

To analyse a pluralist approach to classical, intuitionist & paraconsistent/relevant logics from a properly proof-first perspective.

[This continues the work of my paper, "Pluralism & Proofs" Erkenntnis 2014.]

J. Michael Dunn

1941–2021



Philosophical Perspectives, 7, Language and Logic, 1993

# Star and Perp: Two Treatments of Negation[1]

J. Michael Dunn
Departments of Philosophy and Computer Science

**1. Star and Perp.** In the literature on non-classical logics there are two common treatments of negation, illustrated by the following semantic clauses:

$(\neg *)\ \ \chi \vDash \neg\varphi$ *iff* $\chi^* \nvDash \varphi$
$(\neg\bot)\ \ \chi \vDash \neg\varphi$ *iff* $\forall\alpha(\alpha \vDash \varphi$ *implies* $\alpha \perp \chi)$.

The first uses a unary operation * ("star") on some underlying set of "states" ("worlds," "situations," "set-ups," "cases," whatever). The second definition uses a binary relation $\perp$ ("perp") on such an underlying set.[2] It is the purpose of this paper to show that there is a close connection between these two apparently different treatments.

The definition $(\neg *)$ is perhaps most famous from the Routley-Meyer semantics for relevance logic (see *e.g.*, Routley and Routley (1972), Routley and Meyer (1973)), though its mathematical essence can be traced to the Białynicki-Birula and Rasiowa (1957) representation of De Morgan lattices (*cf*. Dunn (1966, 1967, 1986).[3] The definition $(\neg\bot)$ is perhaps most famous from the Goldblatt (1974) semantics for orthologic, though its most familiar current use is in the Girard (1987) semantics for linear logic. It too has a more ancient history, going back to Birkhoff (1941) in his example of a Galois connection as determined by a "polarity," defined using an arbitrary binary relation. This in turn generalizes the orthogonality operator on closed subspaces of a Hilbert space. K. Došen (1986) should also be recommended for a treatment of various negations in the neighborhood of the intuitionistic one, but with semantics done in the perp style.[4]

My main interest in the relationship between the two treatments of negation is motivated by the fact that the perp definition is the one that falls right out of the general "gaggle theoretic" considerations of Dunn (1990) about how to define semantical conditions for $n$-placed logical operators using $n$ + 1-placed accessibility relations,[5] and yet it is often convenient to understand the De

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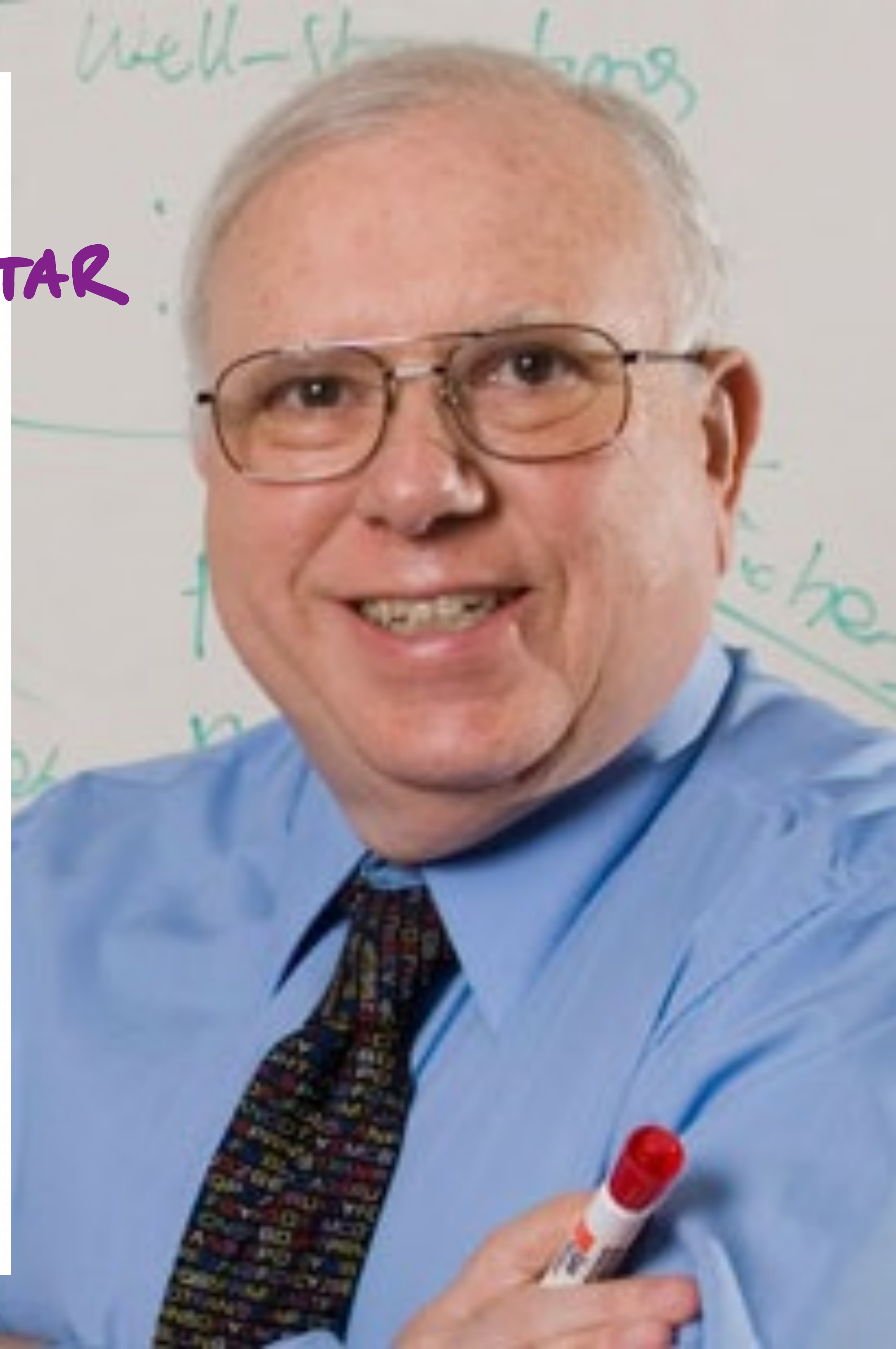

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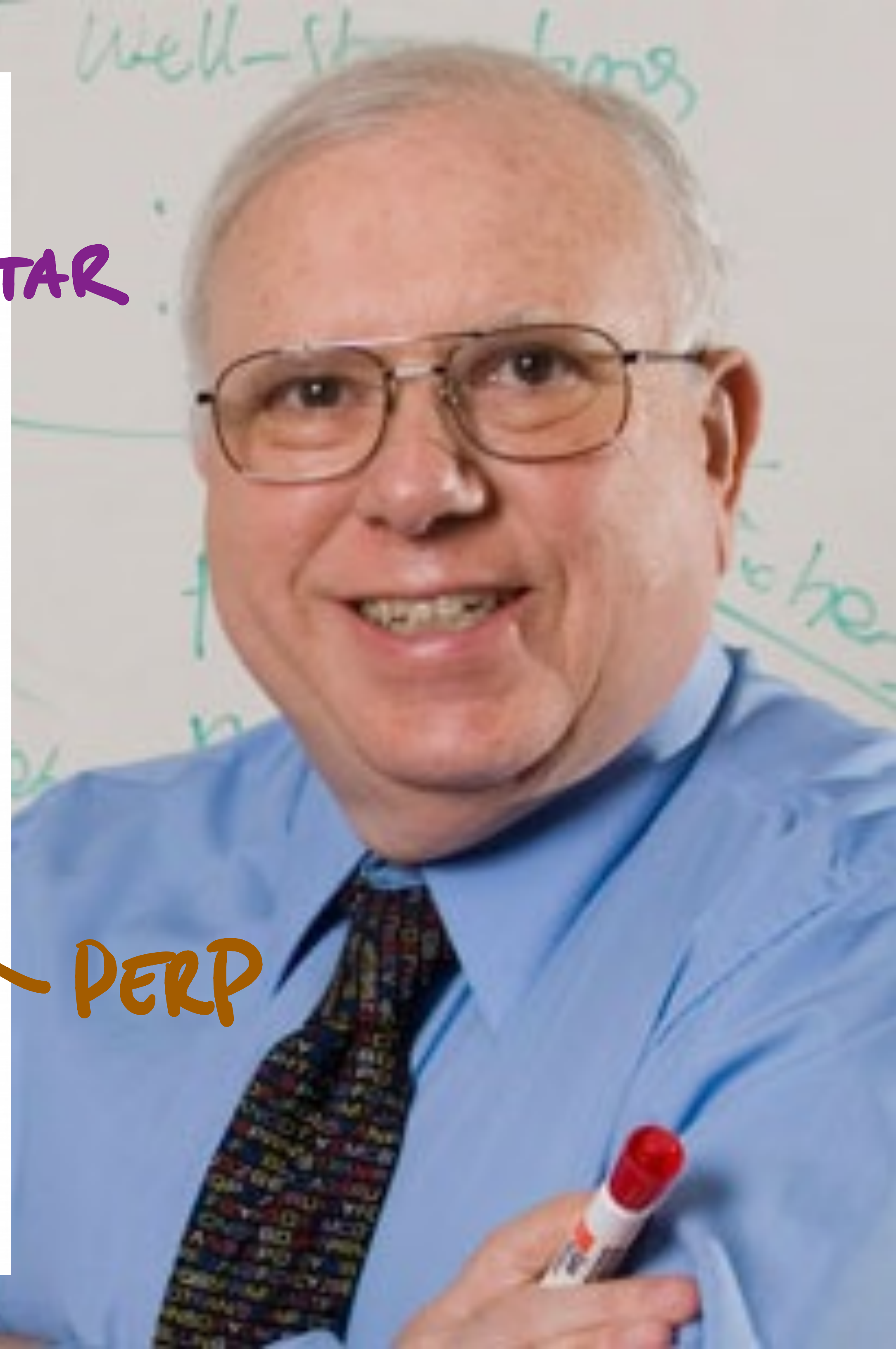


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# Negation on the Australian Plan

Francesco Berto[1,2] · Greg Restall[3]

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**Abstract**
We present and defend the Australian Plan semantics for negation. This is a comprehensive account, suitable for a variety of different logics. It is based on two ideas. The first is that negation is an exclusion-expressing device: we utter negations to express incompatibilities. The second is that, because incompat*ibility* is modal, negation is a modal operator as well. It can, then, be modelled as a quantifier over points in frames, restricted by accessibility relations representing compatibilities and incompatibilities between such points. We defuse a number of objections to this Plan, raised by supporters of the American Plan for negation, in which negation is handled via a many-valued semantics. We show that the Australian Plan has substantial advantages over the American Plan.

**Keywords** Negation · Compatibility semantics · Kripke semantics · Non-classical logics · Many-valued logics · Modal logics

1. Natural Deduction
2. Proofs & speech acts in context
3. Negation & $\perp$
4. Plurality

1. Natural Deduction

2. Proofs & speech acts in context

3. Negation & $\perp$

4. Plurality

$$\neg p \wedge \neg q \succ \neg(p \vee q)$$

$$\dfrac{\dfrac{[p \vee q]^3 \qquad \dfrac{\dfrac{\neg p \wedge \neg q}{\neg p}\,\wedge E \qquad [p]^1}{\bot}\,\neg E \qquad \dfrac{\dfrac{\neg p \wedge \neg q}{\neg q}\,\wedge E \qquad [q]^2}{\bot}\,\neg E}{\bot}\,\vee E^{2,3}}{\neg(p \vee q)}\,\neg I^3$$

Proofs from X to A meet a justification request for A from a position in which the members of X are granted.

$$\dfrac{\begin{matrix} \vdots & & \vdots \\ A & & B \end{matrix}}{A \wedge B}\ \wedge I \qquad \dfrac{\begin{matrix} \vdots \\ A \wedge B \end{matrix}}{A}\ \wedge E \qquad \dfrac{\begin{matrix} \vdots \\ A \wedge B \end{matrix}}{B}\ \wedge E$$

$$\dfrac{\begin{matrix} \vdots \\ A \end{matrix}}{A \vee B}\ \vee I \qquad \dfrac{\begin{matrix} \vdots \\ B \end{matrix}}{A \vee B}\ \vee I \qquad \dfrac{\begin{matrix} & [A]^{i} & [B]^{j} \\ \vdots & \vdots & \vdots \\ A \vee B & C & C \end{matrix}}{C}\ \vee E^{i,j}$$

$$\dfrac{\begin{matrix}\vdots\\ \neg A\end{matrix} \qquad \begin{matrix}\vdots\\ A\end{matrix}}{\bot}\ \neg E
\qquad\qquad
\dfrac{\begin{matrix}[A]^{i}\\ \vdots\\ \bot\end{matrix}}{\neg A}\ \neg I
\qquad\qquad
\dfrac{\begin{matrix}\vdots\\ \bot\end{matrix}}{A}\ \bot E$$

?

Think of a "proof of $\bot$" as a refutation of the assumptions, rather than a proof of a special kind of statement.

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array} \qquad \cancel{A}}{\bot}$$

$$\dfrac{\begin{array}{c}[\cancel{A}]\\ \vdots\\ \bot\end{array}}{A}$$

This is a mild bilateralism.

Proofs from a background where claims are ruled <u>in</u> & others ruled <u>out</u>.

$$\dfrac{\begin{matrix}\vdots\\ A\end{matrix} \qquad \cancel{A}}{\bot}\uparrow \qquad\qquad \dfrac{\begin{matrix}[\cancel{A}]\\ \vdots\\ \bot\end{matrix}}{A}\downarrow$$

This is a mild bilateralism.

Proofs from a background where claims are ruled in & others ruled out.

$X ; Y \succ A$, or equivalently, $X \succ A ; Y$

$$\frac{X;Y \succ A}{X;A,Y \succ \bot}$$

$$\frac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \cancel{A}}{\bot}\;\uparrow \qquad\qquad \frac{\begin{matrix}[\cancel{A}] \\ \vdots \\ \bot\end{matrix}}{A}\;\downarrow$$

$$\frac{X;A,Y \succ \bot}{X;Y \succ A}$$

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Proofs from a background where claims are ruled <u>in</u> & others ruled <u>out</u>.

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$$\dfrac{X \succ A;Y}{X \succ\ ;A,Y}$$

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array} \qquad \cancel{A}}{\bot}\ \uparrow$$

$$\dfrac{\begin{array}{c}[\cancel{A}]\\ \vdots\\ \bot\end{array}}{A}\ \downarrow$$

$$\dfrac{X;A,Y \succ \bot}{X;Y \succ A}$$

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$$\frac{\bot}{A}\,\bot E$$

(⊥E turns out to just be the vacuous case of ↓, which unifies two different kinds of irrelevance in one phenomenon.)

$$\begin{array}{c}[\cancel{A}]\\ \vdots \\ \dfrac{\bot}{A}\,\downarrow\end{array}$$

$$\frac{X \succ\ ;\ Y}{X \succ A\ ;\ Y}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(p\wedge q) \qquad \dfrac{[p]^2 \quad [q]^1}{p\wedge q}\,{\scriptstyle \wedge I}}{\bot}\,{\scriptstyle \neg E}}{\neg q}\,{\scriptstyle \neg I^1}}{\neg p \vee \neg q}\,{\scriptstyle \vee I} \qquad [\cancel{\neg p\vee\neg q}]^3}{\bot}\,{\scriptstyle \uparrow}}{\neg p}\,{\scriptstyle \neg I^2}}{\neg p\vee\neg q}\,{\scriptstyle \vee I} \qquad [\cancel{\neg p\vee\neg q}]^3}{\bot}\,{\scriptstyle \uparrow}}{\neg p\vee\neg q}\,{\scriptstyle \downarrow^3}
$$

$$\neg(p\wedge q) \succ \neg p \vee \neg q$$

What is distinctive about constructive proof?

How can we understand the classical / intuitionistic boundary?

$$\dfrac{\dfrac{\neg\neg p \qquad \dfrac{\dfrac{[p]^1 \quad [\cancel{p}]^2}{\bot}\uparrow}{\neg p}\neg I^1}{\bot}\neg E}{p}\downarrow^2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]^1}{p \vee \neg p}\vee I \qquad [\cancel{p \vee \neg p}]^2}{\bot}\uparrow}{\neg p}\neg I^1}{p \vee \neg p}\vee I \qquad [\cancel{p \vee \neg p}]^2}{\bot}\uparrow}{p \vee \neg p}\downarrow^2$$

$\neg\neg p \vdash p$

$\vdash p \vee \neg p$

To reason constructively, you could avoid the denial rules.

But... why?

Two forms of denial.

strong

weak.

Two forms of denial.

strong

weak

It is natural to understand this in terms of a scope distinction.

inner

outer

$(\neg A)\omega$

$\neg(A\omega)$

$\omega$ is a context of some kind

Let's take contexts to be warrants,

as seems appropriate, constructively.

$$\frac{\neg A\ \omega \qquad A\ \omega}{\perp} \qquad\qquad \frac{A\ \omega \qquad \cancel{A\ \omega}}{\perp}$$

$$\frac{\begin{matrix}[A\ \omega] \\ \vdots \\ \perp\end{matrix}}{\neg A\ \omega}$$

$$\frac{\neg A_{\omega} \qquad A_{\omega}}{\bot} \quad ✓ \qquad\qquad \frac{A_{\omega} \qquad \cancel{A_{\omega}}}{\bot} \quad ✓$$

$$\frac{\begin{array}{c}[A_{\omega}] \\ \vdots \\ \bot\end{array}}{\neg A_{\omega}}$$

$$\frac{\neg A^{\omega} \qquad A^{\omega}}{\perp}$$ ✓

$$\frac{A^{\omega} \qquad \cancel{A^{\omega}}}{\perp}$$ ✓

$$\frac{\begin{matrix}[A^{\omega}] \quad \cancel{A^{\omega}} \\ \vdots \\ \perp\end{matrix}}{\neg A^{\omega}}$$ ✗

Not in the presence of negative assumptions.

Let's step back...

1. Natural Deduction
2. Proofs & speech acts in context
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warrant

$A \; x$

warrant

$A \quad x$

modal/epistemic alternatives

warrant

$A$ $x$

modal/epistemic alternatives

Subject matter,
situations,
scenarios

# Modal Reasoning

$$\dfrac{\dfrac{\dfrac{\Box p \; {}^{1}}{p \; {}^{2}}\;\Box E \qquad \dfrac{\Box q \; {}^{1}}{q \; {}^{2}}\;\Box E}{p \wedge q \; {}^{2}}\;\wedge I}{\Box(p \wedge q) \; {}^{1}}\;\Box I$$

$$\frac{\Box A \quad i}{A \quad j}\ \Box E \qquad\qquad \frac{\begin{array}{c}\vdots\\ A \quad i\end{array}}{\Box A \quad j}\ \Box I$$

(here, $i$ is appropriately arbitrary – e.g. not present in the assumptions)

We can understand the 'world' labels thickly (in terms of a prior understanding of worlds), or thinly, as discourse markers.

Let's grant that $P$ is necessary. Now, suppose things had gone differently. Then (since $P$ is necessary) regardless, we have $P$.

1

Let's grant that $p$ is necessary. Now, suppose things had gone differently. Then (since $p$ is necessary) regardless, we have $p$.

2

$$\dfrac{\dfrac{\neg\neg p \qquad \dfrac{\dfrac{[p]^1 \quad [\cancel{p}]^2}{\bot}\uparrow}{\neg p}\neg I^1}{\bot}\neg E}{p}\downarrow^2$$

Classical proof from $\neg\neg p$ to $p$.

$$\dfrac{\dfrac{\neg\neg p \qquad \dfrac{\dfrac{[p]^1 \quad [\cancel{p}]^2}{\perp}\uparrow}{\neg p}\,\neg I^1}{\perp}\,\neg E}{p}\,\downarrow^2$$

Classical proof from $\neg\neg p$ to $p$.

$$\dfrac{\dfrac{\neg\neg p\,w \qquad \dfrac{\dfrac{[p\,w]^1 \quad [\cancel{p\,w}]^2}{\perp}\uparrow}{\neg p\,w}\,\neg I^1}{\perp}\,\neg E}{p\,w}\,\downarrow^2$$

Breaks down here.

The $\neg I$ step is classically acceptable, but not so, constructively.

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# NEGATION in Contexts

... more generally

$$\neg A\, x \qquad A\, y$$

# Negation in Contexts

... more generally

$$\frac{\neg A\,x \qquad A\,y}{x \perp y}\ \neg E$$

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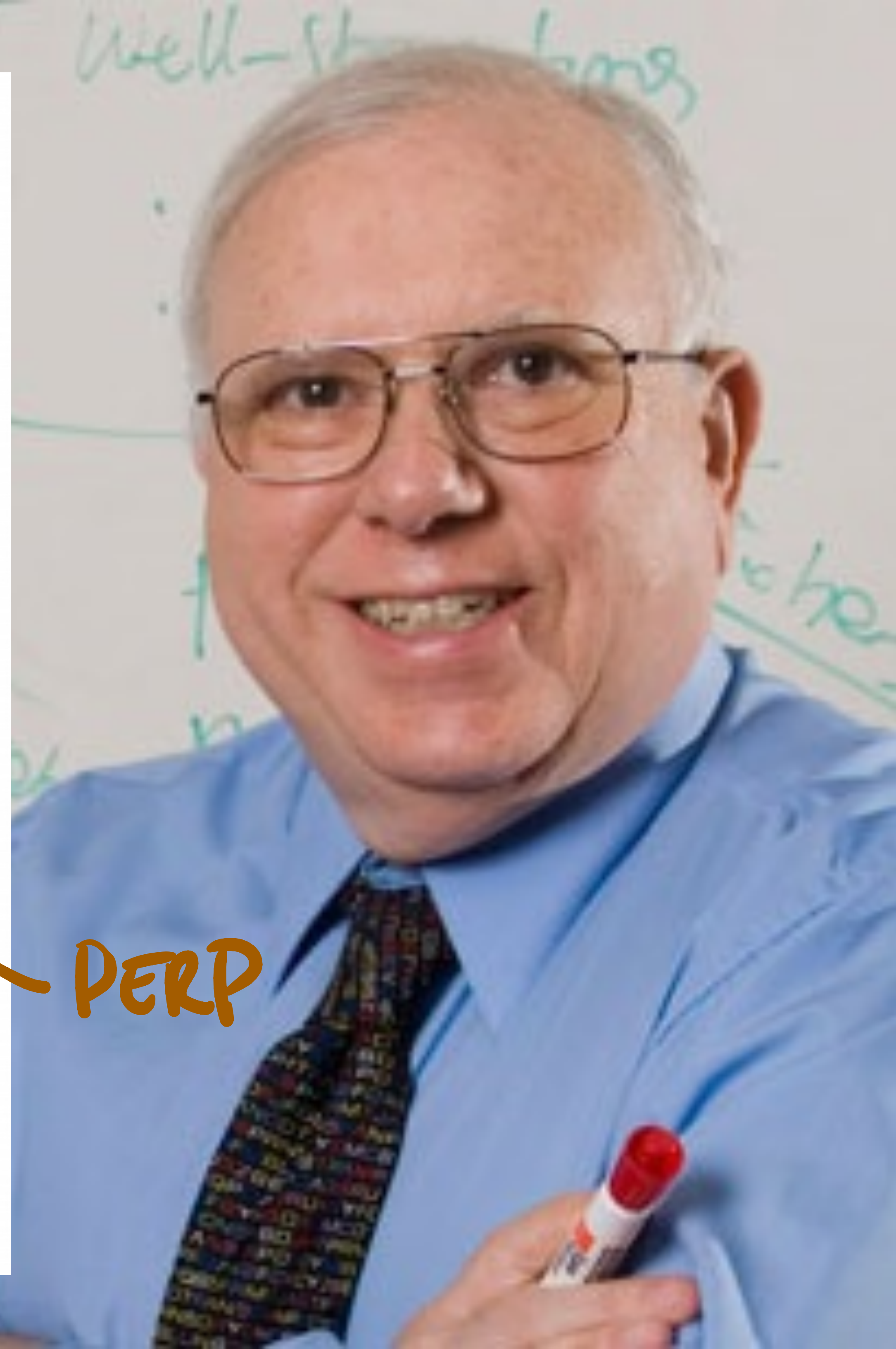

# Negation in Contexts

... more generally

$$\frac{\neg A x \qquad A y}{x \perp y}\ \neg E$$

What might a matching $\neg I$ rule look like?

# Invert ¬E !

$$\frac{\neg A x \qquad A y}{x \perp y}\ \neg E$$

$$\dfrac{\begin{array}{c}[A y]\\ \Pi \\ x \perp y\end{array}}{\neg A x}\ \neg I$$

where $y$ is free in $\Pi$.
(ie $\Pi[y := z]$ is a proof of $x \perp z$ from $A z$ & the same other assumptions as used in $\Pi$.)

# NORMALISING

$$\dfrac{\dfrac{\begin{array}{c}[A\ y]\\ \Pi\\ x \perp y\end{array}}{\neg A\ x}\ \neg I \qquad \begin{array}{c}\Pi'\\ A\ z\end{array}}{x \perp z}\ \neg E \qquad \rightsquigarrow \qquad \begin{array}{c}\Pi'\\ A\ z\\ \Pi[y := z]\\ x \perp z\end{array}$$

$$
\dfrac{
  \neg p \vee \neg q \;\; x
  \qquad
  \dfrac{
    [\neg p \;\; x]^1
    \qquad
    \dfrac{[p \wedge q \;\; y]^3}{p \;\; y}\,\wedge E
  }{x \perp y}\,\neg E
  \qquad
  \dfrac{
    [\neg q \;\; x]^2
    \qquad
    \dfrac{[p \wedge q \;\; y]^3}{q \;\; y}\,\wedge E
  }{x \perp y}\,\neg E
}{
  \dfrac{x \perp y}{\neg(p \wedge q) \;\; x}\,\neg I^3
}\,\vee E^{1,2}
$$

$p \vdash \neg\neg p$ ??

$$\dfrac{\dfrac{[\neg p]^1 \qquad p}{\bot}\;\neg E}{\neg\neg p}\;\neg I^1$$

$p \vdash \neg\neg p$ ??

$$\dfrac{\dfrac{[\neg p]^1 \qquad p}{\bot}\,\neg E}{\neg\neg p}\,\neg I^1$$

$$\dfrac{\dfrac{\dfrac{[\neg p\; y]^1 \qquad p\; x}{y \perp x}\,\neg E}{x \perp y}\,???}{\neg\neg p\; x}\,\neg I^1$$

$p \vdash \neg\neg p$ ??

$$\cfrac{\cfrac{[\neg p]^1 \qquad p}{\bot}\,\neg E}{\neg\neg p}\,\neg I^1$$

$$\cfrac{\cfrac{\cfrac{[\neg p\, y]^1 \qquad p\, x}{y \perp x}\,\neg E}{x \perp y}\,\perp\text{-Symmetry}}{\neg\neg p\, x}\,\neg I^1$$

$P \vdash \neg\neg P$ ??

$$\cfrac{\cfrac{[\neg P]^1 \qquad P}{\perp}\,\neg E}{\neg\neg P}\,\neg I^1$$

$$\cfrac{\cfrac{\cfrac{[\neg P\, y]^1 \qquad P\, x}{y \perp x}\,\neg E \qquad I\, x}{x \perp y}\,\text{Invert}}{\neg\neg P\, x}\,\neg I^1$$

$P, \neg P \vdash \perp$ ??

$$\dfrac{\dfrac{\neg P \; x \qquad P \; x}{x \perp x}\;\neg E}{\perp}\;\text{Consistency}$$

$P, \neg P \vdash \perp$ ??

$$\cfrac{\cfrac{\neg P \quad x \qquad\qquad P \quad x}{x \perp x \qquad\qquad \text{Cons} \quad x}\ \neg E}{\perp}\ \text{Cons.}$$

$\neg\neg p \vdash p$ ??

This is more difficult to prove using the $\bot$ rules.

$$\cfrac{\cfrac{\neg\neg p \qquad \cfrac{\cfrac{[p]^1 \quad [\cancel{p}]^2}{\bot}\,\uparrow}{\neg p}\,\neg I^1}{\bot}\,\neg E}{p}\,\downarrow^2$$

$\neg\neg p \vdash p$ ??

It is easy if we use $*$ instead.

$$\frac{\neg A\, x^{*} \qquad A\, x}{\perp}\ \neg E^{*}$$

$$\frac{\begin{array}{c}[A\, x]\\ \vdots \\ \perp\end{array}}{\neg A\, x^{*}}\ \neg I^{*}$$

$$\frac{A\, x}{A\, x^{**}}\ {**}+$$

$$\frac{A\, x^{**}}{A\, x}\ {**}-$$

$$\dfrac{\dfrac{\dfrac{\neg\neg P\,x}{\neg\neg P\,x^{**}}\;{**}^{+} \qquad \dfrac{\dfrac{[P\,x]^{1}\quad [\cancel{P\,x}]^{2}}{\bot}\,\uparrow}{\neg P\,x^{*}}\,\neg I^{1}}{\bot}\,\neg E}{P\,x}\,\downarrow^{2}$$

There is a lot we can do with tagged proofs of this form, and classical, intuitionistic, & paraconsistent/relevant logics are well-suited to this framework.

1. Natural Deduction

2. Proofs & speech acts in context

3. Negation & $\perp$

4. Plurality

**Option 1** Unique best match between everyday deductive reasoning & a proof formalism.

(eg: untagged proofs, tagged proofs, with denial, without, etc...)

Option 1 Unique best match between everyday deductive reasoning & a proof formalism,

1a ∴ one unique best set of rules, relative to that formalism.

1b ∴ a plurality of sets of rules possible, formalising different norms of deductive proof.

## Option 1

Unique best match between everyday deductive reasoning & a proof formalism,

1a ~~& one unique best set of rules, relative to that formalism.~~

NOT REALLY PLURALIST, or only very weakly so.

1b & a plurality of sets of rules possible, formalising different norms of deductive proof.

Option 1 Unique best match between everyday deductive reasoning & a proof formalism,

1a ~~& one unique best set of rules, relative to that formalism.~~

NOT REALLY PLURALIST, or only very weakly so.

1b & a plurality of sets of rules possible formalising different norms of deductive proof.

THIS OPTION IS PLURALIST

## Example 1: Untagged proofs

$$\dfrac{\dfrac{\neg\neg p \qquad \dfrac{\dfrac{[p]^1 \quad [\cancel{p}]^2}{\perp}\uparrow}{\neg p}\,\neg I^1}{\perp}\,\neg E}{p}\,\downarrow^2$$

The $\neg I$ step is classically acceptable but not so, constructively.

## Example 2: Tagged proofs

$$\dfrac{\dfrac{\neg p \;\; x \qquad p \;\; x}{x \perp x}\,\neg E \qquad \dfrac{}{\mathrm{Cons}\; x}\,\mathrm{Cons}I}{\perp}\,\mathrm{Cons}E$$

Consistency holds constructively, but not so in a paraconsistent logic.

You can model all the way up to classical logic from the $\perp$ foundation with two extra primitives

$$\frac{\mathrm{Cons}\,x \qquad x \perp x}{\perp}\ \mathrm{Cons}\,E \qquad\qquad \frac{}{\mathrm{Cons}\,x}\ \mathrm{Cons}\,I$$

$$\frac{\mathrm{Comp}\,x \qquad \cancel{A}x \qquad Ay}{x \perp y}\ \mathrm{Comp}\,E \qquad\qquad \frac{}{\mathrm{Comp}\,x}\ \mathrm{Comp}\,I$$

$$
\begin{prooftree}
\AxiomC{$\neg\neg p$}
\AxiomC{$[p]^1$}
\AxiomC{$[\cancel{p}]^2$}
\RightLabel{$\uparrow$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg p$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\downarrow^2$}
\UnaryInfC{$p$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\neg\neg p\ x$}
\AxiomC{}
\RightLabel{$\mathrm{CompI}$}
\UnaryInfC{$\mathrm{Comp}\ x$}
\AxiomC{$[\cancel{p}\ x]^2$}
\AxiomC{$[p\ y]^1$}
\RightLabel{$\mathrm{CompE}$}
\TrinaryInfC{$x \perp y$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg p\ x$}
\RightLabel{$\neg E$}
\BinaryInfC{$x \perp x$}
\AxiomC{}
\RightLabel{$\mathrm{ConsI}$}
\UnaryInfC{$\mathrm{Cons}\ x$}
\RightLabel{$\mathrm{ConsE}$}
\BinaryInfC{$\bot$}
\RightLabel{$\downarrow^2$}
\UnaryInfC{$p\ x$}
\end{prooftree}
$$

Is one of these better or more deeply <u>logical</u> than the other?

**Option 1** Unique 'best' match between everyday deductive reasoning & a proof formalism.

**Option 2** There is a plurality of levels of formal analysis of everyday deductive reasoning, each of which gets at different norms of deductive proof.

Options 1b & 2 both have virtues.

Both are kinds of pluralisms about proof & deductive logic.

This framework gives us different ways to explore logical pluralism.

# THE UPSHOT

- A FLEXIBLE & UNIFIED proof-theoretic FRAMEWORK encompassing classical, constructive & paraconsistent/relevant logics.

- A NEW ANGLE from which to view the difference between classical, constructive & paraconsistent/relevant validity

- NEW QUESTIONS about whether there is one best level of analysis of "the" structure of a proof.