The Many Uses of Proofs

Philosophy, Language & More

Greg Restall

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June 13, 2022

MY PLAN

1. SUBSTRUCTURAL LOGICS
2. LOGICAL PLURALISM
3. MODELS & PROOFS
4. INTERPRETING THE SEQUENT CALCULUS
5. PROOFS & MODAL REASONING
6. PROOFS & DIALOGUE

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2. LOGICAL PLURALISM
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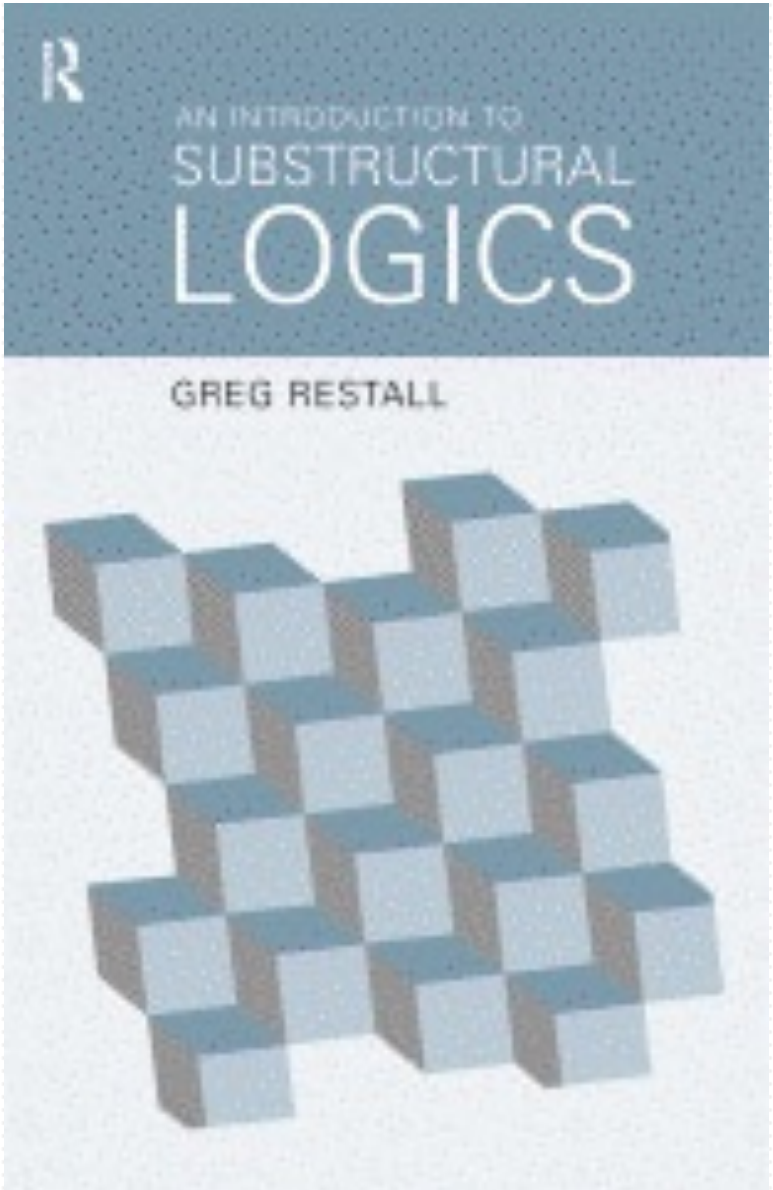


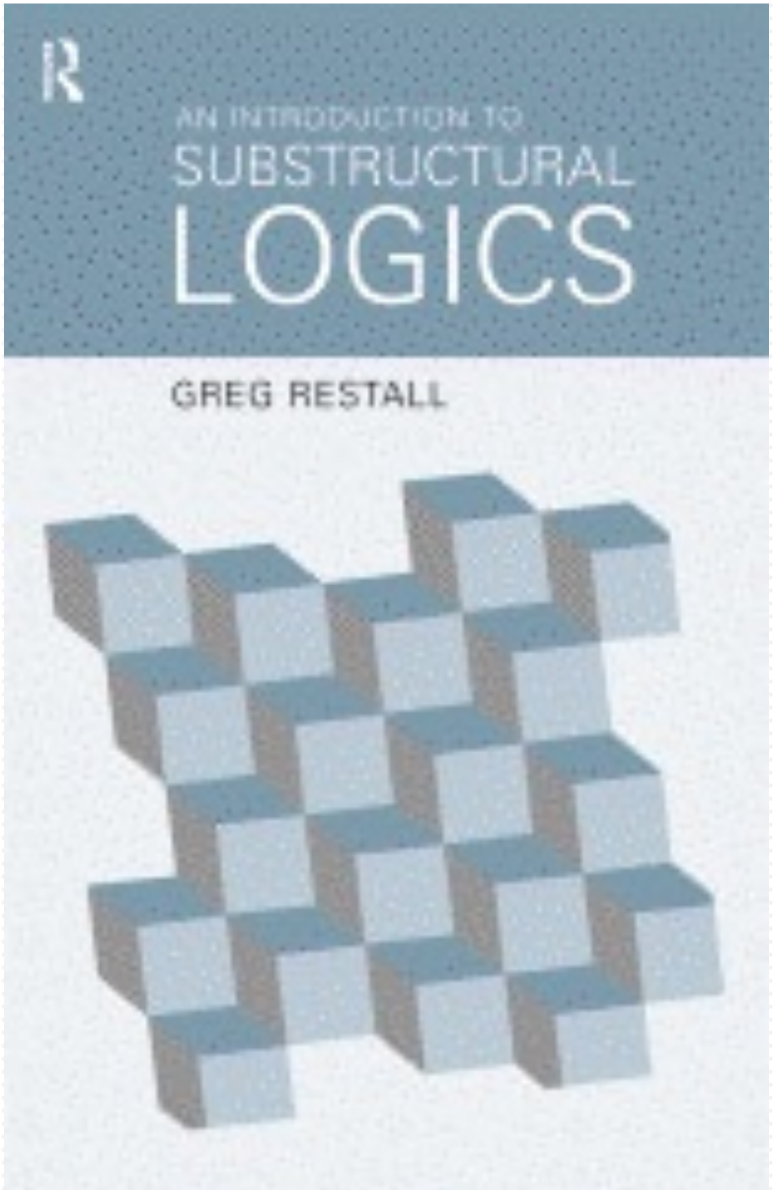
ROUTLEDGE, 2000

RELEVANT LOGICS

LINEAR LOGICS

LAMBEK CALCULUS

⋮



ROUTLEDGE, 2000

RELEVANT LOGICS

LINEAR LOGICS

LAMBEK CALCULUS

⋮

This is a very large collection of logics, with a rich set of family resemblances, & a diverse collection of tools & techniques developed in parallel in different research communities, in philosophy, mathematics, computer science, linguistics, etc...

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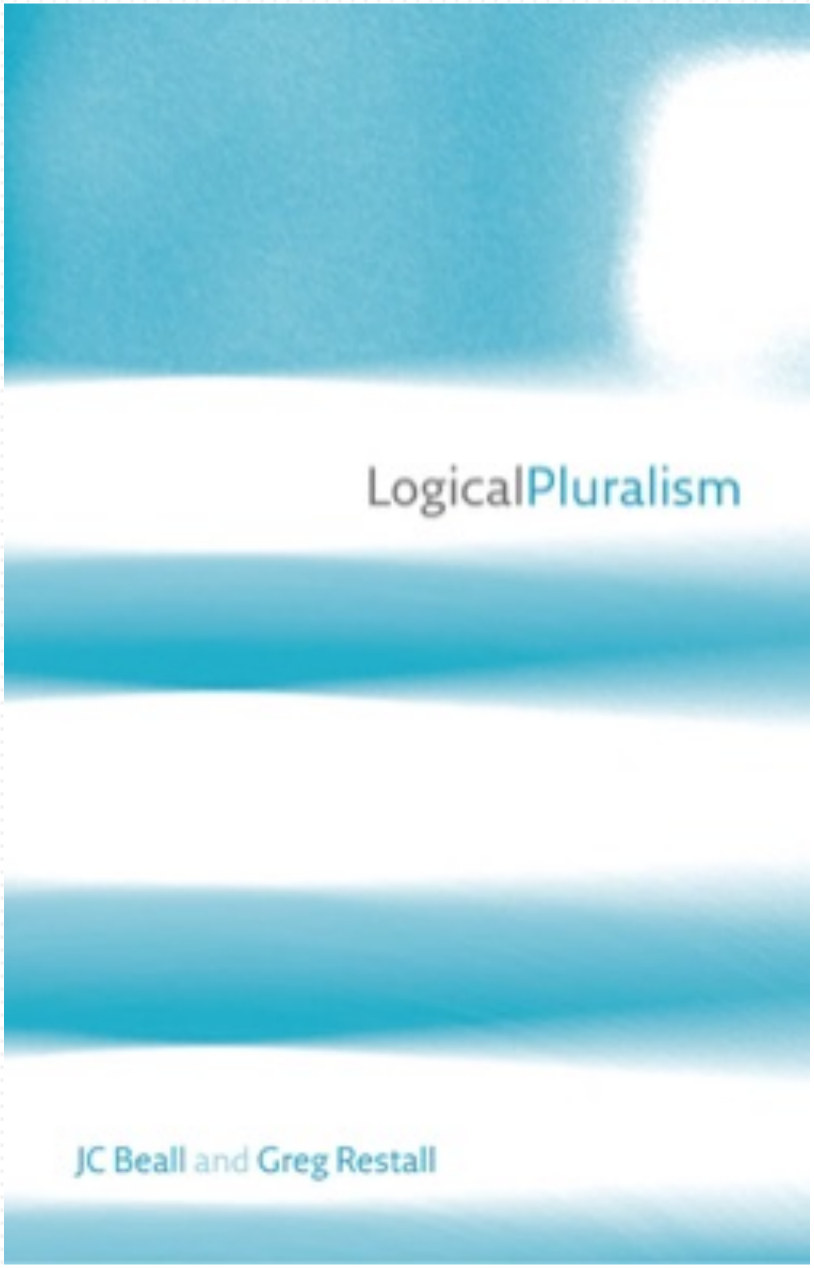
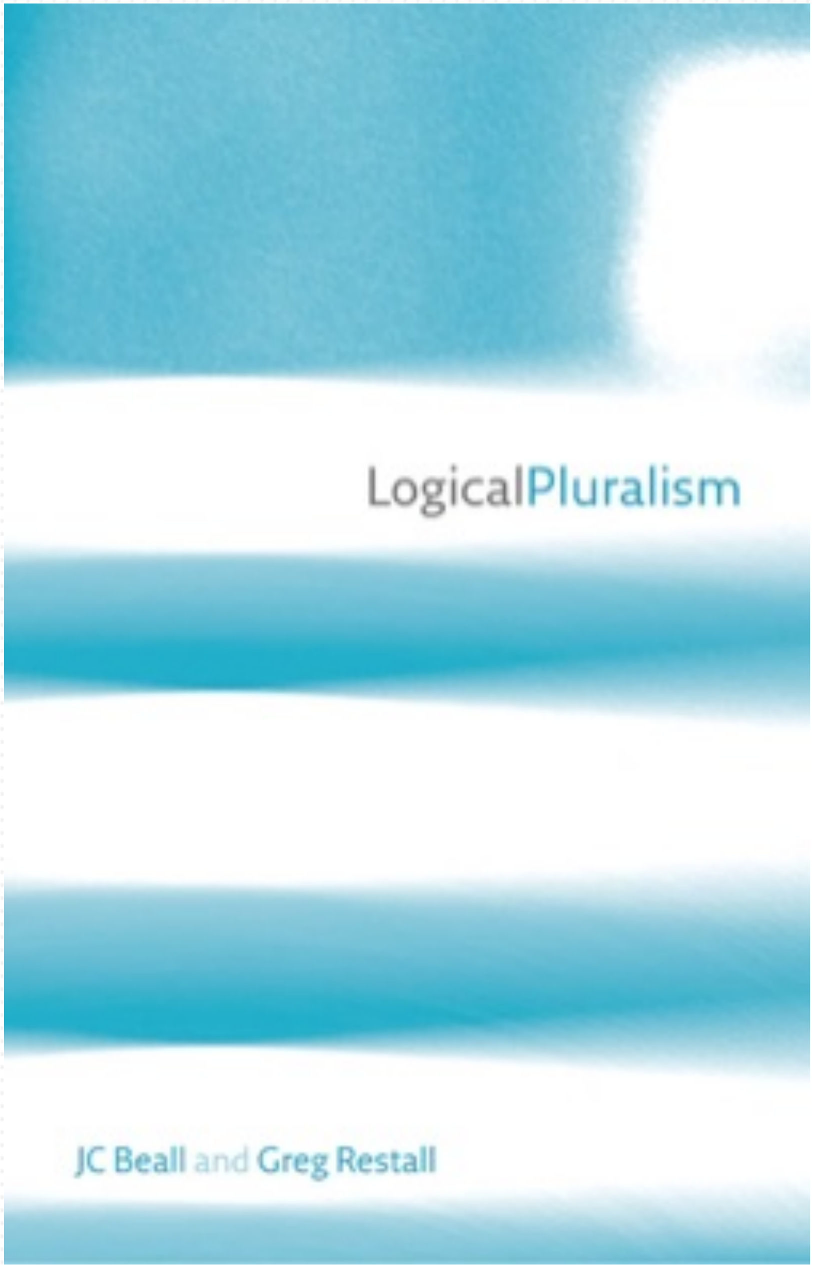


OXFORD U.P. 2006

According to a RELEVANT LOGIC, the argument from $p, \neg p$ to q is INVALID.

According to CLASSICAL (or INTUITIONISTIC) LOGIC, the argument is VALID.



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According to a RELEVANT LOGIC, the argument from $p, \neg p$ to q is INVALID.

According to CLASSICAL (or INTUITIONISTIC) LOGIC, the argument is VALID.

PARTISANS for one or other system argue that the argument is really valid or really invalid. They are LOGICAL MONISTS.

We argued that you don't need to choose — there is more than one sense of deductive validity.

We argued for this model-theoretically.

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A relevant counterexample to an argument, can tell you how the conclusion doesn't relevantly follow from the premises, but this DOES NOT MEAN that it's possible for the premises to be TRUE & the conclusion FALSE.

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A relevant counterexample to an argument can tell you how the conclusion doesn't relevantly follow from the premises, but this DOES NOT MEAN that it's possible for the premises to be TRUE & the conclusion FALSE.

(The same goes for the relationship between CLASSICAL & CONSTRUCTIVE/INTUITIONISTIC validity, we argued.)

1. Substructural Logics
2. Logical Pluralism
3. **Models & Proofs**
4. Interpreting the Sequent Calculus
5. Proofs & Modal Reasoning
6. Proofs & Dialogue

Soundness & Completeness

$$
\begin{tikzcd}
X \vdash A \arrow[rr, bend left] & & X \vDash A \arrow[ll, bend left]
\end{tikzcd}
$$

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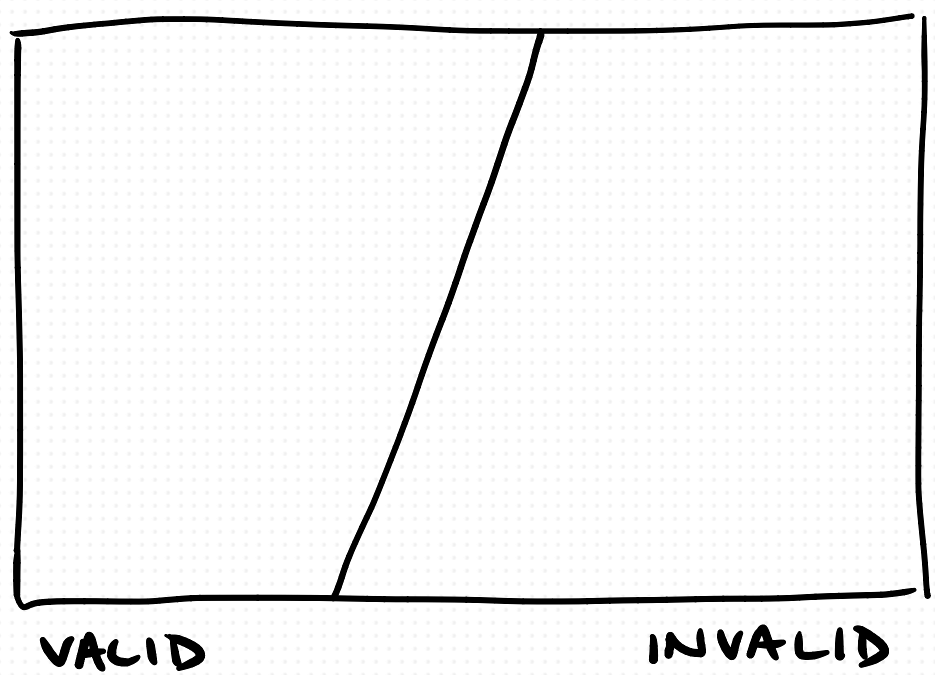
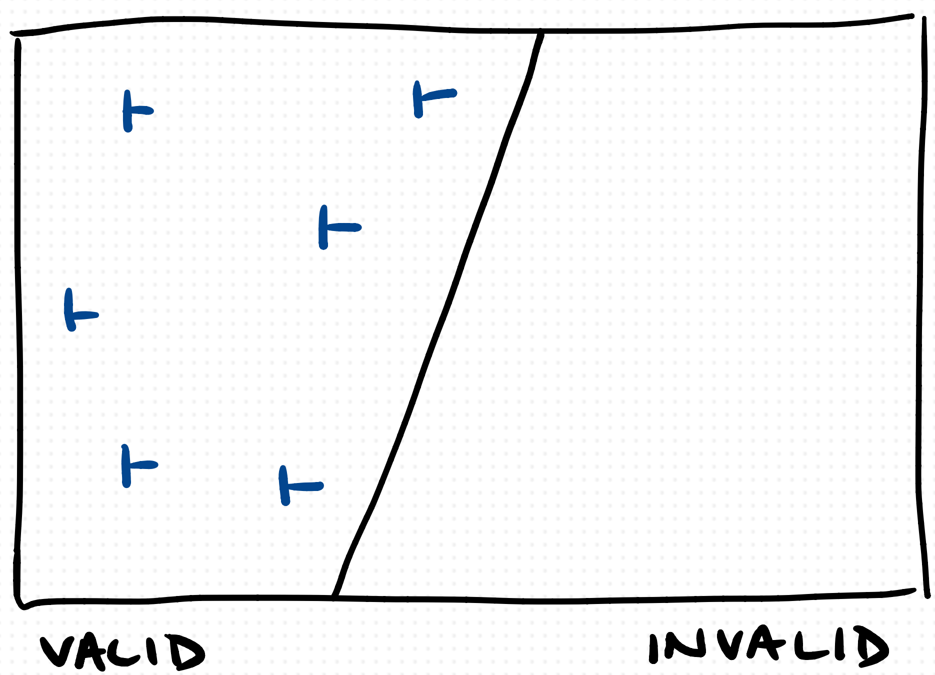


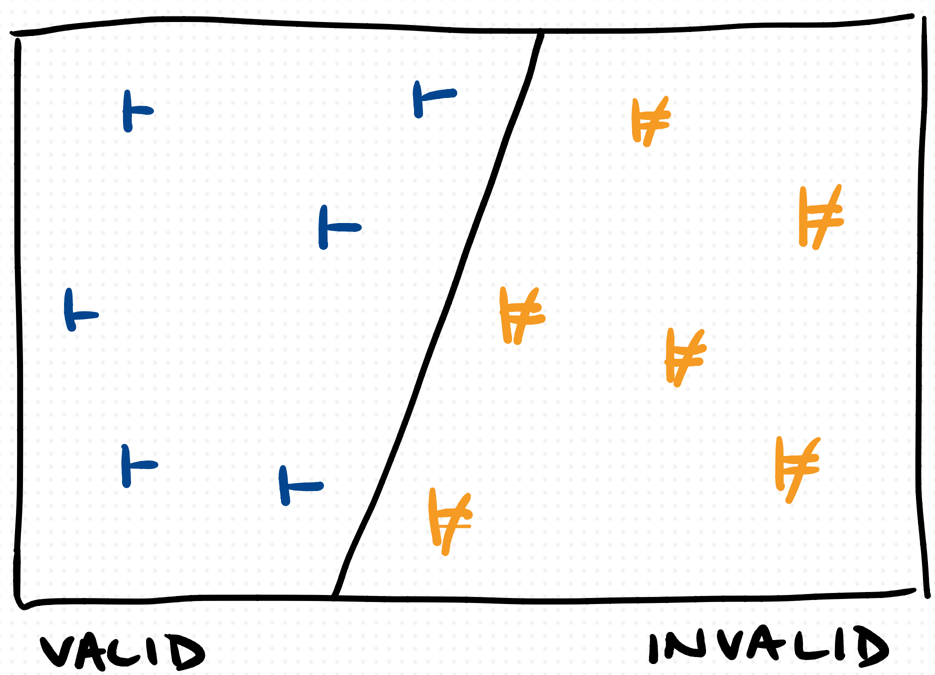
Soundness & Completeness

$$\begin{tikzcd} X \vdash A \arrow[r, bend left] & X \vDash A \arrow[l, bend left] \end{tikzcd}$$



Soundness & Completeness

$$\begin{tikzcd} X \vdash A \arrow[rr, bend left] & & X \models A \arrow[ll, bend left] \end{tikzcd}$$



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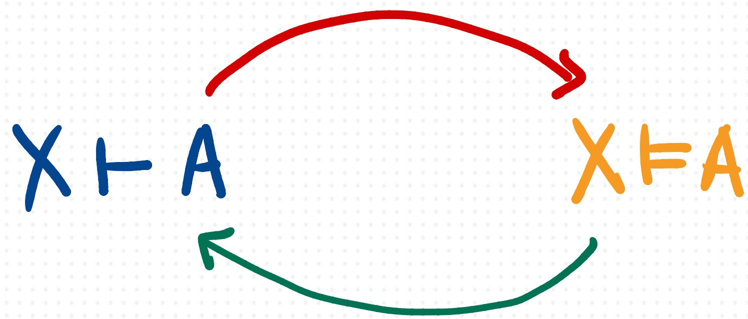
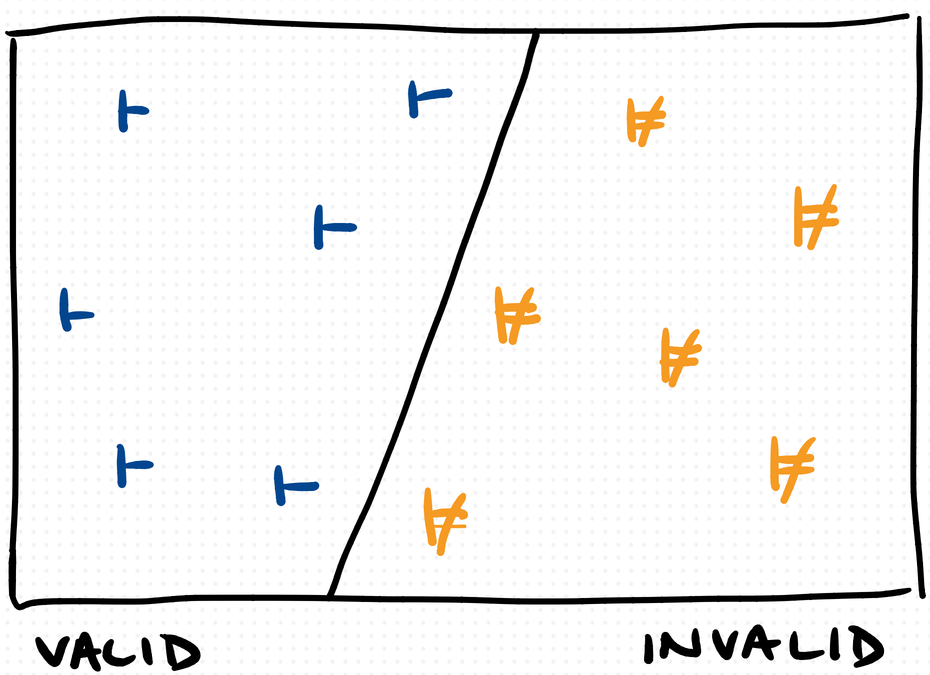




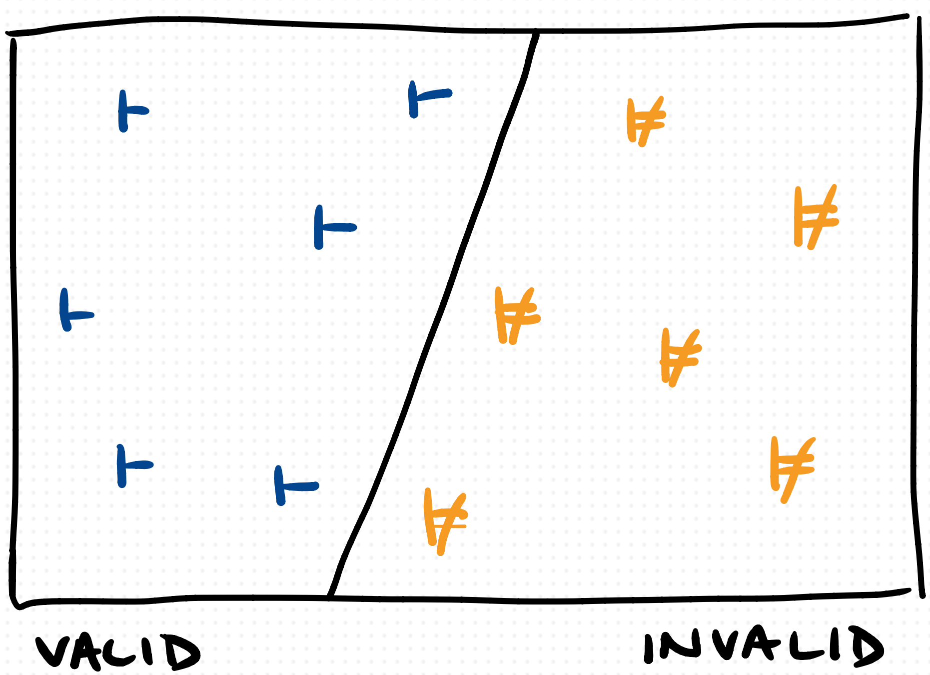
Philosophers & semanticists have focused on using models to interpret a language

Soundness & Completeness

$$\begin{tikzcd} X \vdash A \arrow[r, bend left] & X \vDash A \arrow[l, bend left] \end{tikzcd}$$

Philosophers & semanticists have focused on using models to interpret a language

...and have taken proofs to be 'mere syntax'



$$
\begin{prooftree}
\AxiomC{$(p \to q) \wedge (p \to r)$}
\RightLabel{$\wedge E$}
\UnaryInfC{$p \to q$}
\AxiomC{$[p]^{1}$}
\RightLabel{$\to E$}
\BinaryInfC{q}
\AxiomC{$(p \to q) \wedge (p \to r)$}
\RightLabel{$\wedge E$}
\UnaryInfC{$p \to r$}
\AxiomC{$[p]^{1}$}
\RightLabel{$\to E$}
\BinaryInfC{r}
\RightLabel{$\wedge I$}
\BinaryInfC{$q \wedge r$}
\RightLabel{$\to I^{1}$}
\UnaryInfC{$p \to (q \wedge r)$}
\end{prooftree}
$$

$$
\dfrac{
\dfrac{\dfrac{x:(p\to q)\land(p\to r)}{\mathsf{fst}\;x : p\to q}\land E \qquad [y:p]^{1}}{(\mathsf{fst}\;x)y : q}\to E
\qquad
\dfrac{\dfrac{x:(p\to q)\land(p\to r)}{\mathsf{snd}\;x : p\to r}\land E \qquad [y:p]^{1}}{(\mathsf{snd}\;x)y : r}\to E
}{
\dfrac{\langle(\mathsf{fst}\;x)y,(\mathsf{snd}\;x)y\rangle : q\land r}{\lambda y\langle(\mathsf{fst}\;x)y,(\mathsf{snd}\;x)y\rangle : p\to(q\land r)}\to I^{1}
}\land I
$$

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$$\dfrac{\dfrac{p \vdash p, q}{p, \neg p \vdash q}\,\neg L}{p \wedge \neg p \vdash q}\,\wedge L$$

$$\dfrac{\dfrac{p, q \vdash q}{p \vdash q, \neg q}\,\neg R}{p \vdash q \vee \neg q}\,\vee R$$

$$\dfrac{\dfrac{p \vdash p, q}{p, \neg p \vdash q}\ \neg L}{p \wedge \neg p \vdash q}\ \wedge L$$

$$\dfrac{\dfrac{p, q \vdash q}{p \vdash q, \neg q}\ \neg R}{p \vdash q \vee \neg q}\ \vee R$$

$$\dfrac{\dfrac{p \vdash p \qquad \dfrac{p \vdash p, q}{\vdash p, p \to q}\ {\to} R}{(p \to q) \to p \vdash p}\ {\to} L}{\vdash ((p \to q) \to p) \to p}\ {\to} R$$

$$X \succ Y$$

Asserting everything in X
& denying everything in Y
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$$\frac{X, A, B \succ Y}{X, A \wedge B \succ Y}\ \wedge\text{Df} \qquad \frac{X \succ A, B, Y}{X \succ A \vee B, Y}\ \vee\text{Df} \qquad \frac{X, A \succ B, Y}{X \succ A \to B, Y}\ \to\text{Df} \qquad \frac{X \succ A, Y}{X, \neg A \succ Y}\ \neg\text{Df}$$

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How do we acquire the concepts of possibility & necessity?

$w \Vdash \Box A$ iff for every world v where wRv, $v \Vdash A$.

$w \vDash \Diamond A$ iff for some world v where wRv, $v \Vdash A$

… possible worlds, in the sense of possible states of affairs are not *really* individuals (just as numbers are not *really* individuals).

To say that a state of affairs obtains is just to say that something is the case; to say that something is a possible state of affairs is just to say that something could be the case; and to say that something is the case ‘in’ a possible state of affairs is just to say that the thing in question would necessarily be the case if that state of affairs obtained, i.e. if something else were the case …

We understand ‘truth in states of affairs’ because we understand ‘necessarily’; not *vice versa*.
— “Worlds, Times and Selves” (1969)

We acquire the concept(s) by learning how to work with different kinds of context shifts in our thought & talk.

$$\dfrac{\dfrac{\dfrac{\Box q \succ \;\mid\; p \succ p}{\Box p, \Box q \succ \;\mid\; \succ p}\,\Box L \qquad \dfrac{\Box p \succ \;\mid\; q \succ q}{\Box p, \Box q \succ \;\mid\; \succ q}\,\Box L}{\Box p, \Box q \succ \;\mid\; \succ p \wedge q}\,\wedge R}{\Box p, \Box q \succ \Box(p \wedge q)}\,\Box R$$

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This generalises to structures with more than one kind of context shift (one where we reason about different states of a system — an "objective" modality; and one where the contexts correspond to different states of information — an "epistemic" modality)

A cut-free sequent system for two-dimensional modal logic, and why it matters

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Completeness
Semantics

ABSTRACT

The two-dimensional modal logic of Davies and Humberstone (1980) [3] is an important aid to our understanding the relationship between *actuality*, *necessity* and *a priori knowability*. I show how a cut-free hypersequent calculus for 2D modal logic not only captures the logic precisely, but may be used to address issues in the epistemology and metaphysics of our modal concepts. I will explain how the use of our concepts motivates the inference rules of the sequent calculus, and then show that the completeness of the calculus for Davies–Humberstone models explains why those concepts have the structure described by those models. The result is yet another application of the completeness theorem.

1. Introduction

The 'two-dimensional modal logic' of Davies and Humberstone [3] is an important aid to our understanding the relationship between *actuality*, *necessity* and *a priori knowability*. It is widely used in philosophical discussions of these notions, but it is by no means uncontroversial [2,4,5,7,13]. Models for the logic are well understood. It is a standard modal logic, but instead of evaluating statements at worlds, we double index, and evaluate at pairs of worlds. A holds at $\langle w, v\rangle$ iff from the perspective of w as the actual world, A would have held had v obtained. Then $\Box A$ holds at $\langle w, v\rangle$ if A holds at $\langle w, v'\rangle$ for each different world v' (A is necessary if it holds at every alternative world), $@A$ holds at $\langle w, v\rangle$ iff A holds at $\langle w, w\rangle$ (A is actually the case if it holds back at the actual world), and $\mathsf{APK}\,A$ holds at $\langle w, v\rangle$ iff A holds at $\langle w', w'\rangle$ for each world w' (if A holds in every circumstance considered as actual, it holds, however things could actually be). In these models, we can consider another world as a subjunctive alternative (had things gone differently, that would have been the case) or as an indicative alternative (we might be wrong and that might actually be the case). $\Box$ is the modal logic corresponding to subjunctive alternatives and APK is the modal logic corresponding to indicative alternatives.

The two notions of necessity fall apart rather radically, as you can see with the presence of the actuality operator @. In any model, $p \equiv @p$ is true at each pair $\langle w, w\rangle$. Suppose p is true at $\langle w, w\rangle$, but that p is false at a subjunctive alternative $\langle w, v\rangle$. Now, at $\langle w, v\rangle$, $@p$ is still true (since p holds at $\langle w, w\rangle$) so at $\langle w, v\rangle$, $p \equiv @p$ is *false*. It follows that it is not *necessary* (in the sense of $\Box$) back at $\langle w, w\rangle$. $\Box(p \equiv @p)$ fails at $\langle w, w\rangle$.

However, there is another sense in which $p \equiv @p$ is 'necessary'—we need not know anything about the nature of the world (in particular, we need not know anything about the truth or falsity of p) to know that $p \equiv @p$ is, in fact, true. At whatever world w we choose to evaluate $p \equiv @p$, we have $p \equiv @p$ true at $\langle w, w\rangle$. In this sense (at any *indicative* alternative), $p \equiv @p$ is true. It is knowable *a priori*. $\mathsf{APK}\,(p \equiv @p)$ is a theorem of two-dimensional modal logic, while $\Box(p \equiv @p)$ is not.

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This generalises to structures with more than one kind of context shift (one where we reason about different states of a system — an "objective" modality; and one where the contexts correspond to different states of information — an "epistemic" modality) and the logic has models of the usual "possible worlds" kind, but the proof rules correspond to capacities of human reasoners.

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MaP
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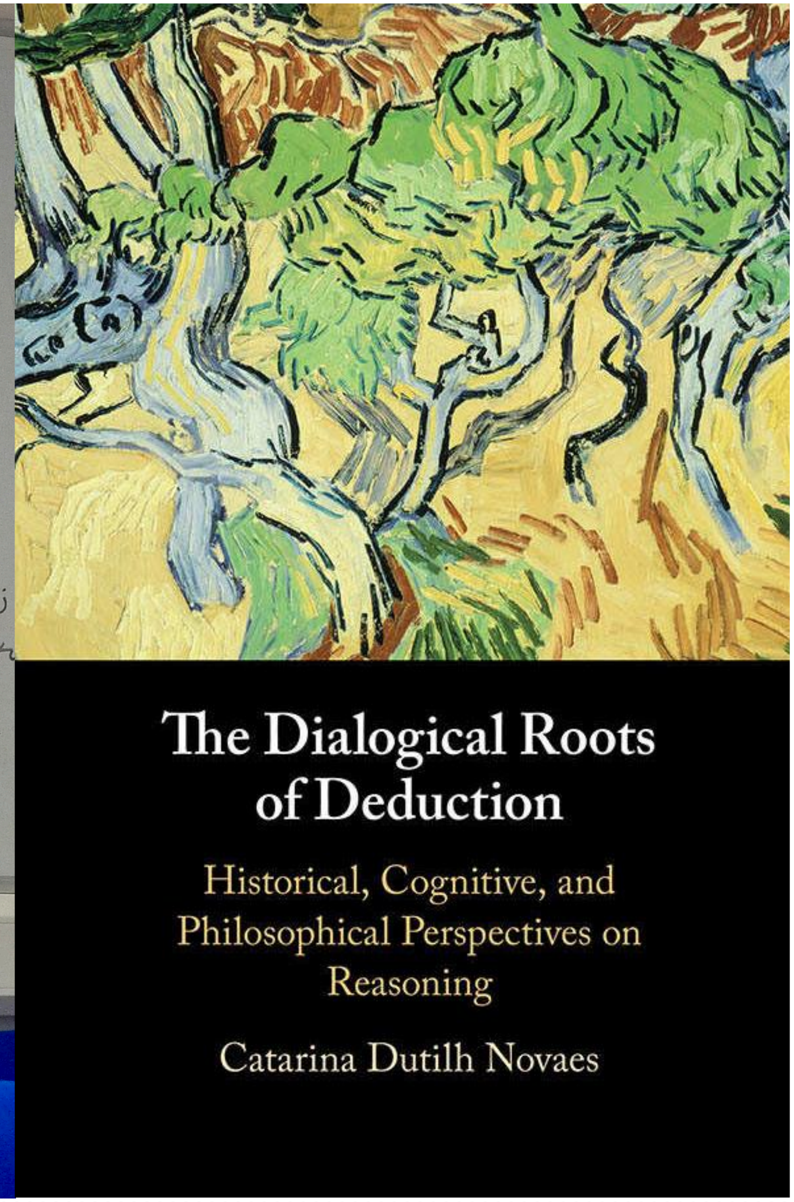
The Dialogical Roots of Deduction
Historical, Cognitive, and Philosophical Perspectives on Reasoning
Catarina Dutilh Novaes

ing, once we have concluded that part of our reasoning—that is, when we have discharged the justification request.

Let's spell this out with a longer example, showing how the different structural and connective rules correspond to dialogue steps and shifts in the common ground. Start with this focussed derivation of the classical sequent $\succ ((p \to q) \to p) \to p$, *Peirce's Law*.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p, q}{\succ p, p \to q}{\scriptstyle \to Df} \qquad \dfrac{(p \to q) \to p \succ (p \to q) \to p}{(p \to q) \to p, p \to q \succ p}{\scriptstyle \to Df}}{(p \to q) \to p \succ p, p}{\scriptstyle Cut}}{(p \to q) \to p \succ p}{\scriptstyle W}}{\succ ((p \to q) \to p) \to p}{\scriptstyle \to Df}$$

We use this derivation to guide a dialogue, with an aim to meet a justification request for an assertion of an instance of Peirce's Law. Here is one example:

ELOISE: $((p \to q) \to p) \to p$.

ABELARD: *Really*? I can never understand conditionals that are deeply left-associated. Why on earth would that be true?

ELOISE: Let's grant $(p \to q) \to p$. I'll now show p.

ABELARD: OK, granted.

CG: *The common ground is now* $[(p \to q) \to p : \;]$.

ELOISE: To show p, let's first rule it out, and if we can show p *then*, it follows regardless.

ABELARD: If you think that'll help, I'll let you grant it. (It seems like ruling p out would make it harder to prove, not easier.)

CG: $[(p \to q) \to p : p]$

ELOISE: Now, given that p is ruled out, we can prove $p \to q$, since if we also rule q out, we have a refutation of p.

ABELARD: I grant that.

CG: $[(p \to q) \to p, p \to q : p]$ (That was using $p \succ p, q$, and then discharging the p, the left branch of the derivation, using $\to$*Df*.)

What goes for this derivation can go for *any* focussed derivation. It follows that we have answers to our original worries about the relationship between the multiple conclusion classical sequent calculus and proofs and *inference*.

- If we understand a *conclusion* of a *proof* as the formula under the focus in a focused sequent derivation, then we both have an answer to the puzzle of Achilles and the Tortoise, by explaining how the failure to follow along such a derivation is the failure to take one of the steps in a derivation as a *definition*. We also can also see why the conclusion in this sense is *single*.
- Since both assertions and denials can be the target of a justification request, this single conclusion can be in the *right* or the *left* of a sequent.
- The making of an *inference* is a (possibly preemptive) answer to a justification request.
- A derivation of a sequent $X \succ A, Y$ $[X, A \succ Y]$ can be transformed into a *procedure* for meeting a justification request for an assertion of A [denial of A] in any available position, appealing only what is granted in $[X : Y]$, and to the defining rules used in that derivation.

But we can do more than answer *those* original concerns. This characterisation of defining rules also have a clearer grasp of the value of derivations, and the role of proof in expanding our knowledge. Having a proof allows us to *do* something that we cannot do without it.

- The bounds, by themselves, can transcend our grasp.
- Is [PA : GC] out of bounds? Who knows? The bounds in the language of first order predicate logic are undecidable.
- Derivations provide one way we can *grasp* complex bounds and *police* them.
- The *negative* view of the bounds is seen in the clash between assertion and denial, and the *positive* view of the bounds is found in the answers we can give to justification requests. *Both* have their role in characterising norms governing our speech acts.
- Adopting *defining rules* is one way to be *very* precise about the norms governing the concepts so defined, combining *safety* (we

There is much to be done on the connections between

- Moves in dialogue
- conceptions of the common ground
- Semantic/pragmatic connections with the c.g. & interpreting accommodation, anaphora, focus, etc.
- Proof structures that exploit these resources.

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- Moves in dialogue
- conceptions of the common ground
- Semantic/pragmatic connections with the c.g. & interpreting accommodation, anaphora, focus, etc.
- Proof structures that exploit these resources.

Of course, the connection between proofs & programs provides another host of opportunities to explore the implementation of all this.....

Thank You!

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