What do we mean?

SEMANTICS, PRACTICES & PLURALISM

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Arché M&L Seminar

3 July 2024

WARNING

This is all pretty fresh.

I'm trying these ideas out with an aim to present them to a general philosophical audience.

(Helpful) feedback is encouraged!

THE ISSUE

AN ANALOGY

THE CLAIM

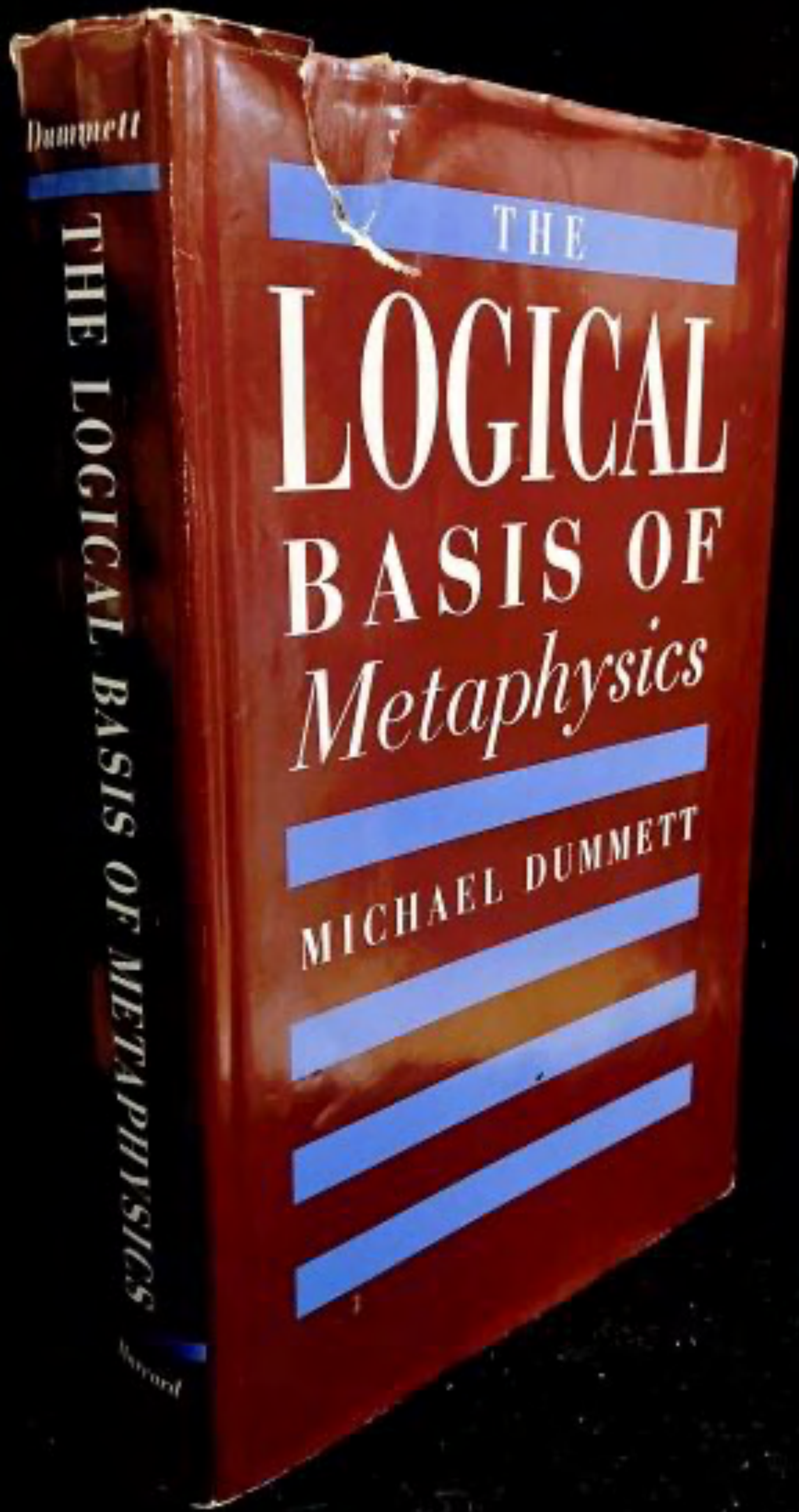
SOME CONSEQUENCES

THE ISSUE

AN ANALOGY

THE CLAIM

SOME CONSEQUENCES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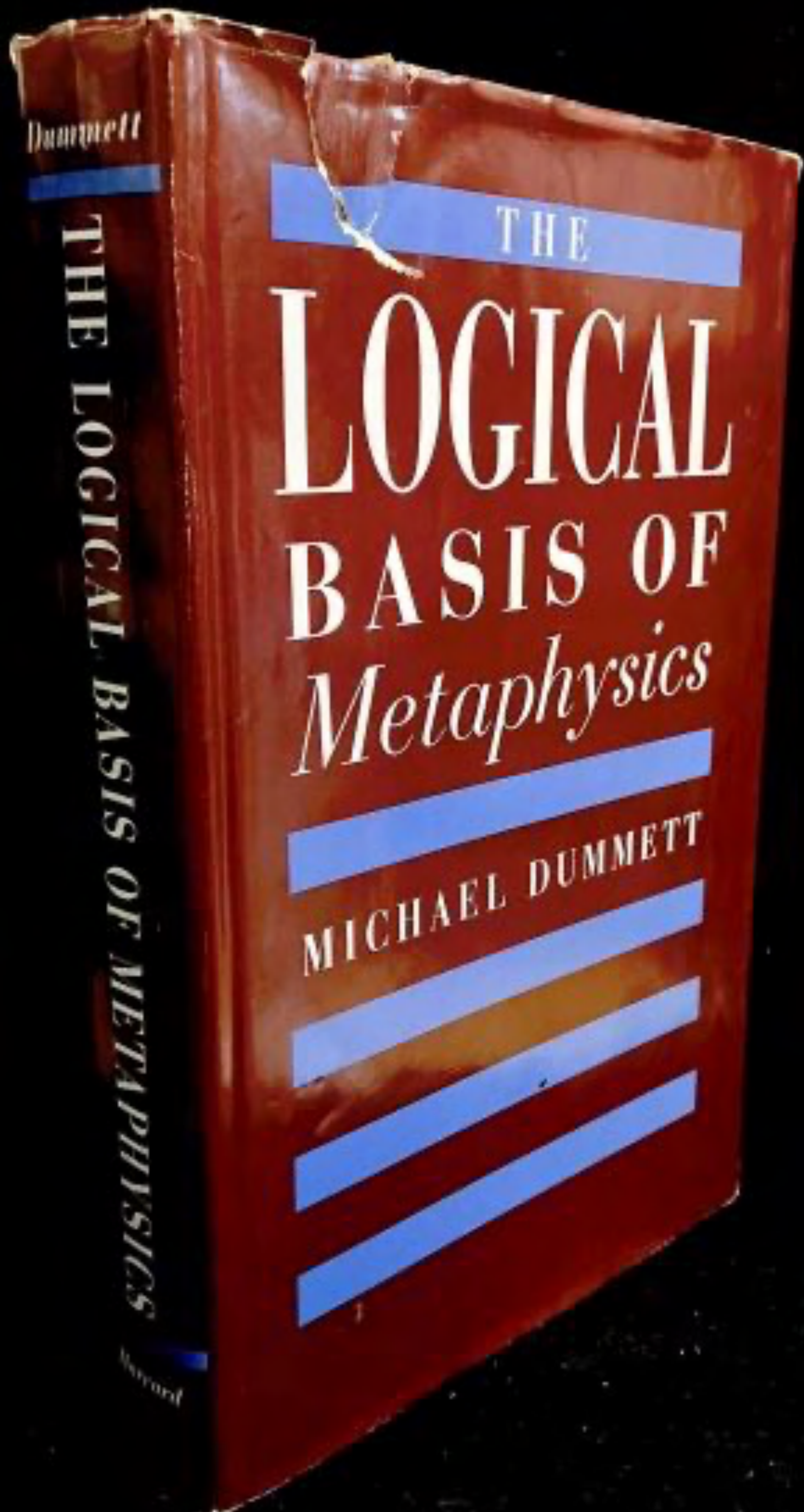
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RT Cook - Philosophy Compass, 2010 - Wiley Online Library
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G Russell, C Blake-Turner - 2013 - plato.stanford.edu
… Much current work on the subject was sparked by a series of papers by JC Beall and Greg Restall (Beall & Restall 2000, 2001; Restall 2002), which culminated in a book (Beall …
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J Van Benthem - The Australasian Journal of **Logic**, 2008 - ojs.victoria.ac.nz
… One is **logical pluralism**, locating the new scope of **logic** in … mind about the crux of what **logic** should become. I would now … And **logical** systems should deal with a wide variety of these, …
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G **Restall** - Erkenntnis, 2014 - Springer
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Carnap's tolerance, meaning, and **logical pluralism**
G **Restall** - The Journal of Philosophy, 2002 - JSTOR
… In some ways, this is a claim that the concept of **logical** consequence is ambiguous. Our **logical pluralism** is more than this, however: we have argued that the core notion of **logical** …
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Logical pluralism
A Paseau - 2007 - JSTOR
… This leads me to deeper concerns about Beall and **Restall's pluralism**. As explained, they …
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Why is $p \vee \neg p$ true?

$$
\begin{prooftree}
\AxiomC{$[\neg(p \vee \neg p)]^2$}
\AxiomC{$[p]^1$}
\RightLabel{$\vee I$}
\UnaryInfC{$p \vee \neg p$}
\RightLabel{$\neg E$}
\BinaryInfC{$\#$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg p$}
\RightLabel{$\vee I$}
\UnaryInfC{$p \vee \neg p$}
\AxiomC{$[\neg(p \vee \neg p)]^2$}
\RightLabel{$\neg E$}
\BinaryInfC{$\#$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\neg\neg(p \vee \neg p)$}
\RightLabel{DNE}
\UnaryInfC{$p \vee \neg p$}
\end{prooftree}
$$

CONSTRUCTIVE PROOF

CLASSICAL, but not CONSTRUCTIVE

RULES USED

CONSTRUCTIVE RULES

$$\frac{A}{A \vee B}\ \vee I \qquad \frac{B}{A \vee B}\ \vee I \qquad \frac{\neg A \quad A}{\#}\ \neg E \qquad \frac{\begin{matrix}[A]^i \\ \vdots \\ \#\end{matrix}}{\neg A}\ \neg I^i$$

CLASSICAL RULES

$$\frac{\neg\neg A}{A}\ \text{DNE}$$

How should we think about the relationship between classical & constructive logic?

OPTION 1

Constructive logic is more restrictive than classical logic.

It has higher standards, & so, can prove fewer things.

To see how some of the most basic results of classical analysis lack computational meaning, take the assertion that every bounded nonvoid set A of real numbers has a least upper bound. (The real number b is the *least upper bound* of A if $a \leqq b$ for all a in A, and if there exist elements of A that are arbitrarily close to b.) To avoid unnecessary complications, we actually consider the somewhat less general assertion that every bounded sequence (x_k) of rational numbers has a least upper bound b (in the set of real numbers). If this assertion were constructively valid, we could compute b, in the sense of computing a rational number approximating b to within any desired accuracy; in fact, we could program a digital computer to compute the approximations for us. For instance, the computer could be programmed to produce, one by one, a sequence $((b_k, m_k))$ of ordered pairs, where each b_k is a rational number and each m_k is a positive integer, such that $x_j \leqq b_k + k^{-1}$ for all positive integers j and k, and $x_{m_k} \geqq b_k - k^{-1}$ for all positive integers k. Unless there exists a general method M that produces such a computer program corresponding to each bounded, constructively given sequence (x_k) of rational numbers, we are not justified, by constructive standards, in asserting that each of the se-

Errett Bishop and Douglas Bridges, *Constructive Analysis* (1985)

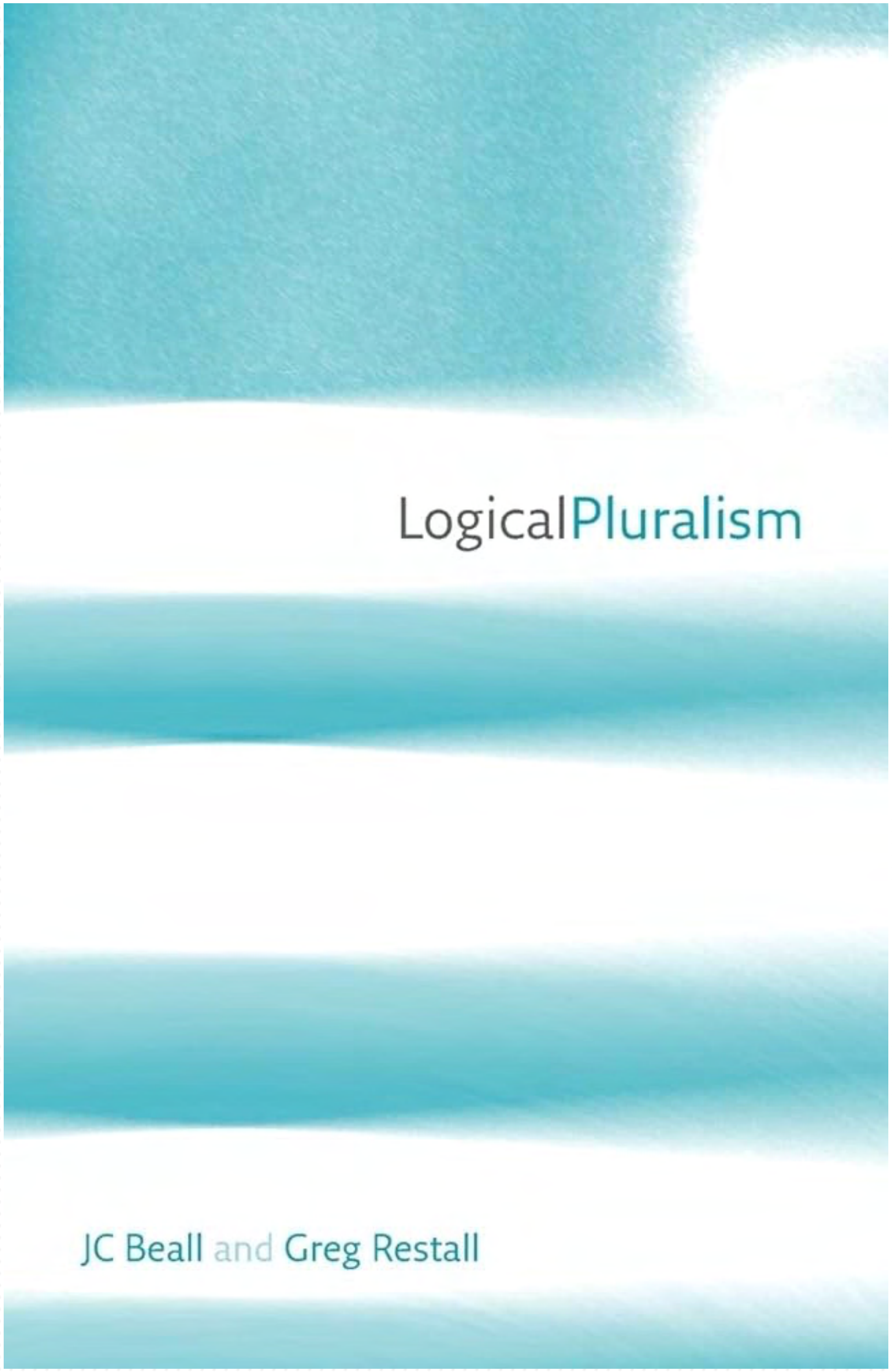
Option 2

Constructive logic is more expressive than classical logic. It identifies fewer statements, & so, has more to say

The law of the excluded middle provides a prime example. Constructively, this principle is not universally valid, as we have seen in Exercise **12.1**. Classically, however, it is valid, because every proposition is either false or not false, and being not false is the same as being true. Nevertheless, classical logic is consistent with constructive logic in that constructive logic does not refute classical logic. As we have seen, constructive logic proves that the law of the excluded middle is positively not refuted (its double negation is constructively true). Consequently, constructive logic is stronger (more expressive) than classical logic, because it can express more distinctions (namely, between affirmation and irrefutability), and because it is consistent with classical logic.

Proofs in constructive logic have computational content: they can be executed as programs, and their behavior is described by their type. Proofs in classical logic also have computational content, but in a weaker sense than in constructive logic. Rather than positively affirm a proposition, a proof in classical logic is a computation that cannot be refuted. Computationally, a refutation consists of a continuation, or control stack, that takes a proof of a proposition and derives a contradiction from it. So a proof of a proposition in classical logic is a computation that, when given a refutation of that proposition derives a contradiction, witnessing the impossibility of refuting it. In this sense, the law of the excluded middle has a proof, precisely because it is irrefutable.

Robert Harper, *Practical Foundations for Programming Languages* (2016)



The LOGICAL PLURALISM of Beall & Restall (2006) takes OPTION 1 & does not consider OPTION 2.

It's time to revisit this issue.

TypeTypology Personal — cs.bham.ac.uk

index | Integers.Type | Rationals.Type | DedekindReals.Type | Groups.Type | CantorSchroederB...

```
TypeTopology

Various new theorems in univalent mathematics written in Agda
--------------------------------------------------------------

Martin Escardo and collaborators,
2010--2024--∞, continuously evolving.
https://www.cs.bham.ac.uk/~mhe/
https://github.com/martinescardo/TypeTopology

Tested with Agda 2.6.4.3

 * Our main use of this development is as a personal blackboard or
   notepad for our research and that of collaborators. In
   particular, some modules have better and better results or
   approaches, as time progresses, with the significant steps kept,
   and with failed ideas and calculations eventually erased.

 * We offer this page as a preliminary announcement of results to be
   submitted for publication, of the kind we would get when we visit
   a mathematician's office.

 * We have also used this development for learning other people's
   results, and so some previously known constructions and theorems
   are included (sometimes with embellishments).

 * The required material on HoTT/UF has been developed on demand
   over the years to fulfill the needs of the above as they arise,
   and hence is somewhat chaotic. It will continue to expand as the
   need arises. Its form is the result of evolution rather than
   intelligent design (paraphrasing Linus Torvalds).

   Our lecture notes develop HoTT/UF in Agda in a more principled
   way, and offers better approaches to some constructions and
   simpler proofs of some (previously) difficult theorems.
   (https://www.cs.bham.ac.uk/~mhe/HoTT-UF-in-Agda-Lecture-Notes/)

   Our philosophy, here and in the lecture notes, is to work with a
   minimal Martin-Löf type theory, and use principles from HoTT/UF
   (existence of propositional truncations, function extensionality,
   propositional extensionality, univalence, propositional resizing)
   and classical mathematics (excluded middle, choice, LPO, WLPO) as
   explicit assumptions for the theorems, or for the modules, that
   require them. As a consequence, we are able to tell very
   precisely which assumptions of HoTT/UF and classical mathematics,
   if any, we have used for each construction, theorem or set of
   results. We also work, deliberately, with a minimal subset of
   Agda. See below for more about the philosophy.

 * There is also a module that links some "unsafe" modules that use
   type theory beyond MLTT and HoTT/UF, which cannot be included in
   this safe-modules index: The system with type-in-type is
   inconsistent (as is well known), countable Tychonoff, and
   compactness of the Cantor type using countable Tychonoff.

   (https://www.cs.bham.ac.uk/~mhe/TypeTopology/AllModulesIndex.html)

 * In our last count, on 2024.06.19, this development has 761 Agda
   files with 215k lines of code, including comments and blank
   lines. But we don't update the count frequently.

Philosophy of the repository
----------------------------

 * We adopt the univalent point of view, even in modules which don't
   assume the univalence axiom. In particular, we take seriously the
   distinction between types that are singletons (contractible),
   propositions, sets, 1-groupoids etc., even when the univalence
   axiom, or its typical consequences such as function
   extensionality and propositional extensionality, are not needed
   to reason about them.

 * We work in a minimal version of intensional Martin-Löf Type
```

A whole lot of mathematics is being done using proof assistants, most of which use constructive logic.

TypeTypology Personal · cs.bham.ac.uk

index | Integers.Type | Rationals.Type | DedekindReals.Type | Groups.Type

```
Martin Escardo, 22nd and 24th January 2020, with further additions
after that.

The Cantor-Schröder-Bernstein for homotopy types, or ∞-groupoids, in Agda
--------------------------------------------------------------------------

An unformalized version of parts of this file was published in
the Journal of Homotopy and Related Structures, Springer, 28th June 2021.
https://doi.org/10.1007/s40062-021-00284-6

There are two parts, which assume function extensionality but not
univalence or the existence of propositional truncations (any
assumption beyond MLTT is explicit in each claim).

(1) A univalent-foundations version of Pierre Pradic and Chad
    E. Brown's argument that Cantor-Schröder-Bernstein implies
    excluded middle in constructive set theory. (Added 22nd January.)
    (https://arxiv.org/abs/1904.09193).

    Their proof, reproduced here, uses the compactness (also known as
    the searchability or omniscience) of ℕ∞.

    (See also Appendix II.)

(2) A proof that excluded middle implies Cantor-Schröder-Bernstein for
    all homotopy types, or ∞-groupoids. (Added 24th January.)

    For any pair of types, if each one is embedded into the other,
    then they are equivalent.

    For this it is crucial that a map is an embedding if and only if
    its fibers are all propositions (rather than merely the map being
    left-cancellable).

    As far as we know, (2) is a new result.

    This part is the Agda version of https://arxiv.org/abs/2002.07079.
    Check our lecture notes to learn HoTT/UF with Agda:
    https://www.cs.bham.ac.uk/~mhe/HoTT-UF.in-Agda-Lecture-Notes/

\begin{code}

{-# OPTIONS --safe --without-K #-}

module CantorSchroederBernstein.CSB where

open import CoNaturals.Type
open import MLTT.Plus-Properties
open import MLTT.Spartan
open import Naturals.Properties
open import NotionsOfDecidability.Decidable
open import TypeTopology.CompactTypes
open import TypeTopology.GenericConvergentSequenceCompactness
open import UF.Base
```

What can we learn from this?

Not just about logic or mathematics, but about meaning, more generally....

THE ISSUE

AN ANALOGY

THE CLAIM

SOME CONSEQUENCES

Consider the relationship between our own practices of COUNTING & CALCULATING, and our use of digital/mechanical aids.

What do we do when we COUNT and CALCULATE?

1 2 3 4 5   1 2 3

• • • • • + • • •

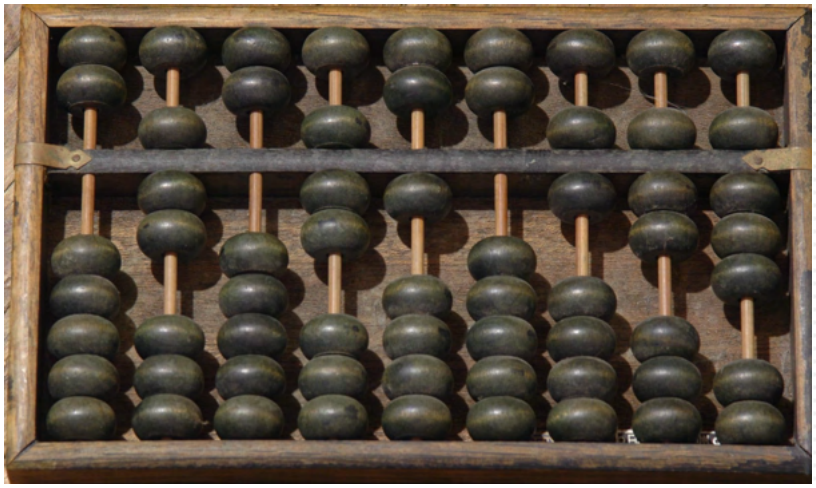
1 2 3 4 5   6 7 8

```
  123
 ×456
 ----
  738
 6150
49200
-----
56088
```

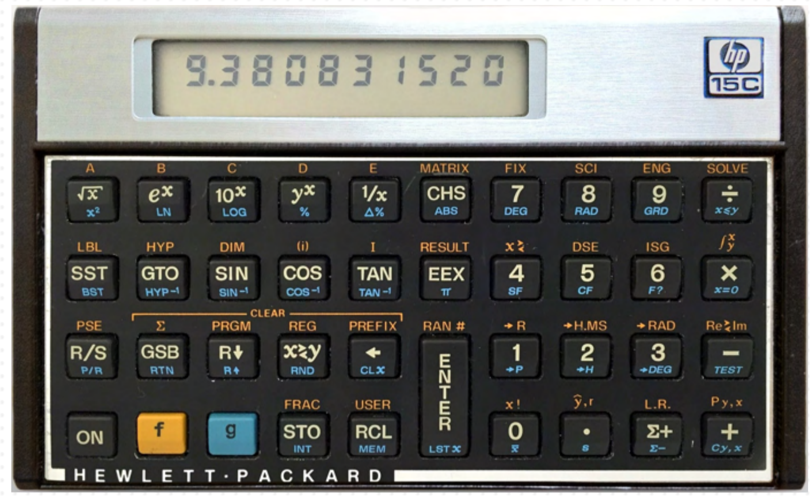
×	1	2	3	4	5
1	1	2	3	4	5
2	6	7	8	9	10
3	11	12	13	14	15

This is a really useful skill!

What about our devices?



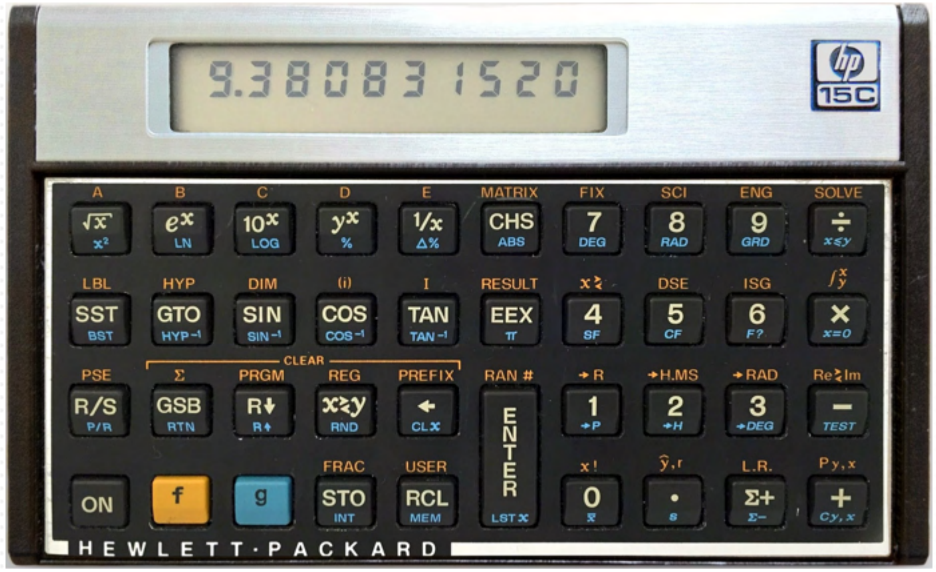
EXTERNALISING MEMORY &
CONSTRAINING REPRESENTATION



... ALL THAT,
AND DOING SOME
CALCULATION, TOO.

Why is this?

What do we need for a device to be able to do this job?



The calculator doesn't count things like we do.

But it is really useful for calculating things, as you'd expect, given the name.

One way to do this is for the calculator to really ~~CALCULATE~~ — to follow some process that (at some level) corresponds to what we do when we calculate.

Ideally, we want the calculator to not only tell us that $f(n)=m$, but give us knowledge that $f(n)=m$.

A Counting Structure

"the number of" $\#F$ predicate

$$\#F = \#G \leftrightarrow \exists f(F \xleftrightarrow{f} G)$$

$$0 = \#\lambda x(x \neq x)$$
$$1 = \#\lambda x(x = 0)$$
$$2 = \#\lambda x(x = 0 \vee x = 1)$$
$$\vdots$$

$$\neg\exists x(Fx \wedge Gx) \to \#(F \vee G) = \#F + \#G$$
$$\vdots$$

Peano Axioms for finite Arithmetic

$$Sx = Sy \to x = y$$
$$0 \neq Sx$$
$$x \neq 0 \to \exists y(x = Sy)$$

$$x + 0 = x$$
$$x + Sy = S(x+y)$$

$$x \times 0 = 0$$
$$x \times Sy = x \times y + x$$

$$[\phi(0) \wedge \forall x(\phi(x) \to \phi(Sx))] \to \forall x\, \phi(x)$$

It is not that we follow any of these rules precisely — they are patterns we can extract from our practice ... and implement.

A machine that implements reasoning along either of these lines is doing arithmetic...

SCHEME 1

$$\#F = \#G \leftrightarrow \exists f(F \overset{f}{\leftrightarrow} G)$$

$$0 = \#\lambda x(x \neq x)$$

$$1 = \#\lambda x(x = 0)$$

$$2 = \#\lambda x(x = 0 \lor x = 1)$$

$$\vdots$$

$$\neg\exists x(Fx \land Gx) \rightarrow \#(F \lor G) = \#F + \#G$$

$$\vdots$$

SCHEME 2

$$Sx = Sy \rightarrow x = y$$

$$0 \neq Sx$$

$$x \neq 0 \rightarrow \exists y(x = Sy)$$

$$x + 0 = x$$

$$x + Sy = S(x + y)$$

$$x \times 0 = 0$$

$$x \times Sy = x \times y + x$$

$$[\phi(0) \land \forall x(\phi(x) \rightarrow \phi(Sx))] \rightarrow \forall x\phi(x)$$

Although they would differ on the details.

(Is there any n where $n = n+1$?

YES for SCHEME 1 but NO for SCHEME 2.)

This makes no difference for everyday calculation, & maybe our counting practice doesn't decide between SCHEME 1 & SCHEME 2.

It would be strange to say that one SCHEME is correct and the other is incorrect.

SCHEME 1

$$\#F = \#G \leftrightarrow \exists f (F \xrightarrow{f} G)$$

$$0 = \#\lambda x (x \neq x)$$

$$1 = \#\lambda x (x = 0)$$

$$2 = \#\lambda x (x = 0 \vee x = 1)$$

$$\vdots$$

$$\neg \exists x (Fx \wedge Gx) \rightarrow \#(F \vee G) = \#F + \#G$$

$$\vdots$$

SCHEME 2

$$Sx = Sy \rightarrow x = y$$

$$0 \neq Sx$$

$$x \neq 0 \rightarrow \exists y (x = Sy)$$

$$x + 0 = x$$

$$x + Sy = S(x+y)$$

$$x \times 0 = 0$$

$$x \times Sy = x \times y + x$$

$$[\phi(0) \wedge \forall x (\phi(x) \rightarrow \phi(Sx))] \rightarrow \forall x \phi(x)$$

Rather, you could say that SCHEME 1 is a theory of cardinal numbers, while SCHEME 2 is a theory of finite ordinals.

(Not that this means we have access to cardinals or ordinals independently of our counting practices.)

[And none of this is to take a stand on what these numbers are, & whether cardinals are ordinals.]

THE UPSHOT

Our everyday counting practice can be explicated in different ways.

These explications can help us understand the different things we can do when we count & calculate, & to implement these practices in machines & programmes.

THE UPSHOT

And if a machine implements calculation using some procedure, then its actions may form part of our grounds for knowledge for some claim, in the same way that our own calculations do.

Let's keep this example in mind....

THE ISSUE

AN ANALOGY

THE CLAIM

SOME CONSEQUENCES

Let's look at one scheme for making some aspects of this practice explicit.

What goes for our COUNTING & CALCULATING practices (& numbers) might also go for our ASSERTING (SUPPOSING, DENYING) & INFERRING practices (& propositions).

& BELIEVING, JUDGING, THINKING, ...

but our emphasis is not on the distinctively cognitive, mental components of judgement.

A proof Π of C from X grounds C in the background X.

CONSTRUCTIVE SCHEME

PROOF

$$\begin{matrix} X \\ \Pi \\ C \end{matrix}$$

REFUTATION

$$\begin{matrix} X \\ \Pi \\ \# \end{matrix}$$

(& STRUCTURAL RULES)

and

$$\dfrac{A \quad B}{A\wedge B}\,\wedge I \qquad \dfrac{A\wedge B}{A}\,\wedge E_1 \qquad \dfrac{A\wedge B}{B}\,\wedge E_2$$

if

$$\dfrac{\begin{matrix}[A]^i\\ \vdots\\ B\end{matrix}}{A\to B}\,\to I^i \qquad \dfrac{A\to B \quad A}{B}\,\to E$$

THESE CONCEPTS ARE RECOGNISABLE AS ANALOGUES OF OURS.

$$\dfrac{A}{A\vee B}\,\vee I_1 \qquad \dfrac{B}{A\vee B}\,\vee I_2$$

$$\dfrac{A\vee B \quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\,\vee E$$

or

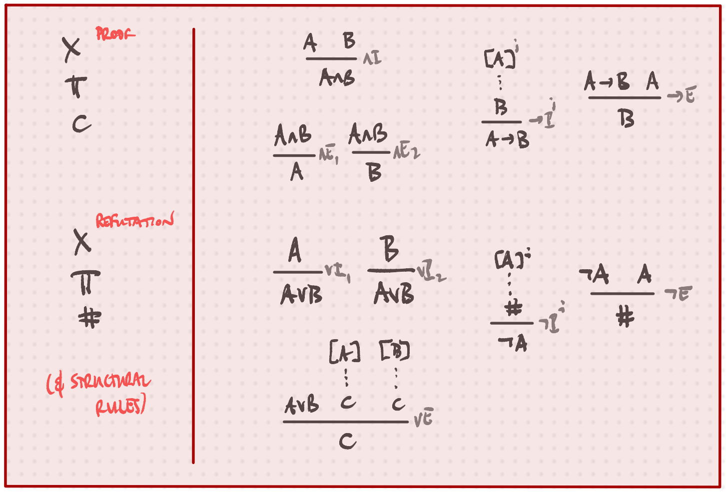
$$\dfrac{\begin{matrix}[A]^i\\ \vdots\\ \#\end{matrix}}{\neg A}\,\neg I^i \qquad \dfrac{\neg A \quad A}{\#}\,\neg E$$

not

A refutation Π of X shows how X is ruled out.

Connective rules act like DEFINITIONS against that background.

CONSTRUCTIVE SCHEME



This SCHEME (& more, involving terms, as well as types, & quantifiers, etc) is implemented in PROOF ASSISTANTS, and is naturally found inside mathematical structures (topological spaces; cartesian closed categories) as an "internal language".

(The Scheme also has the virtue that a proof of $A \vee B$ gives you a means to construct a proof of A or of B, and a proof of $\exists x \phi(x)$ gives you a means to find some n along with a proof of $\phi(n)$.)

What more could be required for a practice organised in this way to be recognised as counting as ASSERTING, INFERRING, & so on?

Well, maybe there is something missing...

DOES THIS SCHEME DO JUSTICE TO OUR USE OF DENIAL?

GADAMER ON LANGUAGE

It is unclear whether there is here a genuine disagreement between Gadamer and Davidson. It is undeniable that someone may lack a concept that others have, and that we now have many concepts that no one had three hundred years ago. New concepts are continually introduced. They cannot always be defined in the existing language, but they can be explained by means of it; a study of how we acquire concepts, such as the concept of infinity, that could not even be expressed before their introduction would be highly illuminating. It is also undeniable that we can now recognize, of certain concepts that were used in some previous age, that they were incoherent or confused. Interpretation of a text requires, not necessarily that we should be able to *express* the concepts it invokes, but that we should be able, in our present language, to *explain* them; and this includes explaining what it was to have those concepts we now regard as confused. Interpretation does not make the heavy demand on the interpreter's stock of concepts that it contain all those invoked in the text (or piece of spoken discourse) that he is interpreting: it makes only the light demand that he be able to explain those concepts, or explain what it is to have them, in his own language. Only if it is impossible to give such an explanation is the interpreter justified in denying that the text has a genuine meaning and expresses no concepts, not even incoherent ones.

It is common usage to take "it is undeniable that..." to be an intensifier.

Michael Dummett
The Nature & Future of Philosophy
(2010)
p. 94

The constructivist takes it to be ==undeniable== that $p \vee \neg p$, since $\neg(p \vee \neg p)$ can be reduced to a contradiction, but they do not take this to amount to a proof of $p \vee \neg p$.

$$
\dfrac{\dfrac{[\neg(p \vee \neg p)]^2 \qquad \dfrac{\dfrac{\dfrac{[\neg(p \vee \neg p)]^2 \qquad \dfrac{[p]^1}{p \vee \neg p}\,{\scriptstyle \vee I}}{\#}\,{\scriptstyle \neg E}}{\neg p}\,{\scriptstyle \neg I^1}}{p \vee \neg p}\,{\scriptstyle \vee I}}{\#}\,{\scriptstyle \neg E}}{\neg\neg(p \vee \neg p)}\,{\scriptstyle \neg I^2}
$$

BILATERALISM — ASSERTION & DENIAL

A ---- claiming, asserting, proposing, supposing A.

$\cancel{A}$ ---- denying A, ruling it out, setting it aside

A --- A? Yes!

$\cancel{A}$ --- A? No!

$$\frac{A \qquad \cancel{A}}{\#}\uparrow \qquad\qquad \frac{\begin{matrix}[\cancel{A}]^{i}\\ \vdots \\ \#\end{matrix}}{A}\downarrow^{i}$$

A holds

A is undeniable.

CLASSICAL SCHEME

PROOF

$$\begin{array}{c} X \\ \pi \\ C \end{array}$$

REFUTATION

$$\begin{array}{c} X \\ \pi \\ \# \end{array}$$

CLASH

$$\frac{A \quad \cancel{A}}{\#}$$

RESOLVE

$$\frac{\begin{array}{c} [\cancel{A}]^k \\ \vdots \\ \# \end{array}}{A}\,{}^k$$

$$\frac{A \quad B}{A \wedge B}\ \wedge I$$

$$\frac{A \wedge B}{A}\ \wedge E_1 \qquad \frac{A \wedge B}{B}\ \wedge E_2$$

$$\frac{\begin{array}{c} [A]^i \\ \vdots \\ B \end{array}}{A \to B}\ \to I^i \qquad \frac{A \to B \quad A}{B}\ \to E$$

$$\frac{A}{A \vee B}\ \vee I_1 \qquad \frac{B}{A \vee B}\ \vee I_2$$

$$\frac{A \vee B \quad \begin{array}{c} [A] \\ \vdots \\ C \end{array} \quad \begin{array}{c} [B] \\ \vdots \\ C \end{array}}{C}\ \vee E$$

$$\frac{\begin{array}{c} [A]^i \\ \vdots \\ \# \end{array}}{\neg A}\ \neg I^i \qquad \frac{\neg A \quad A}{\#}\ \neg E$$

Why is $p\vee\neg p$ true?

$$\dfrac{\dfrac{\dfrac{[\neg(p\vee\neg p)]^2 \quad \dfrac{\dfrac{\dfrac{[\neg(p\vee\neg p)]^2 \quad \dfrac{[p]^1}{p\vee\neg p}\,{\scriptstyle \vee I}}{\#}\,{\scriptstyle \neg E}}{\neg p}\,{\scriptstyle \neg I^1}}{p\vee\neg p}\,{\scriptstyle \vee I}}{\#}\,{\scriptstyle \neg E}}{\neg\neg(p\vee\neg p)}\,{\scriptstyle \neg I^2}}{p\vee\neg p}\,{\scriptstyle DNE}$$

$\rightsquigarrow$

$$\dfrac{\dfrac{[\cancel{p\vee\neg p}]^2 \quad \dfrac{\dfrac{\dfrac{[\cancel{p\vee\neg p}]^2 \quad \dfrac{[p]^1}{p\vee\neg p}\,{\scriptstyle \vee I}}{\#}\,{\scriptstyle \uparrow}}{\neg p}\,{\scriptstyle \neg I^1}}{p\vee\neg p}\,{\scriptstyle \vee I}}{\#}\,{\scriptstyle \uparrow}}{p\vee\neg p}\,{\scriptstyle \downarrow^2}$$

This proof makes sense relative to the background assumptions about assertion & denial.

THE CLAIM

The CONSTRUCTIVE Scheme & the CLASSICAL Scheme are both, to some extent, implicit in our assertoric & inferential practice just as cardinal & ordinal conceptions of number are in our counting practice.

Both are recognisably inferential schemes, and both have their uses in regimenting inference & developing theories.

THE ISSUE

AN ANALOGY

THE CLAIM

SOME CONSEQUENCES

THE SEMANTICS & EPISTEMOLOGY OF PROOF ASSISTANTS

- When we are using hybrid machine/human systems, it is valuable to understand the rules in play & the semantics of the systems (both Machine & Human).

  This involves not only the operational rules (those governing each connective & quantifier etc), but the structural rules governing the space of 'propositions'.

- This will constrain what a proof or disproof can mean.

CONSTRUCTIVE TABOOS

- Unprovable claims like $p \vee \neg p$, $\forall x \phi(x) \vee \exists x \neg \phi(x)$, ... are not taken to be false but a TABOO in intuitionistic mathematics — something outlawed & to be avoided in any properly constructive theory.

- We can see this as either a metatheoretic claim (unprovability), or as a way of expressing a notion of denial that has no object-language correlate.

AND WHAT ABOUT PLURALISM?

- Nothing here settles that issue.

- You could accept both constructive & classical schemes as equally legitimate.
- Or you could reject one or other as incomplete or misguided.

- But I hope that now, at least, the options & the stakes have been somewhat clarified.

Thank You!