To present an account of the nature of proof, with the aim of explaining how proof could actually play the role in reasoning that it does, and answering some long-standing puzzles about the nature of proof.
My Aim

Along the way, I’ll explain how Kreisel’s *squeezing argument* helps us understand the connection between an informal notion of validity and the notions formalised in our accounts of proofs and models, and the relationship between proof-theoretic and model-theoretic analyses of logical consequence.
Outline

Motivation

Background

What Proofs Are & What They Do

Counterexamples & Kreisel’s Squeeze

Consequences for How Proofs Work
MOTIVATION
Every *drink* (in our fridge) is either a *beer* or a *lemonade*.
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\[
(\forall x)(Dx \supset (Bx \lor Lx)).
\]
Example Proof

Every *drink* (in our fridge) is either a *beer* or a *lemonade* 

\((\forall x)(Dx \supset (Bx \lor Lx))\). So either every *drink* is a *beer* or some *drink* is a *lemonade*
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\((\forall x)(Dx \supset (Bx \lor Lx))\). So either every *drink* is a *beer* or some *drink* is a *lemonade* 
\((\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)\).
Every *drink* (in our fridge) is either a *beer* or a *lemonade* 

\[ (\forall x)(Dx \supset (Bx \lor Lx)) \]. So either every *drink* is a *beer* or some *drink* is a *lemonade* \[ (\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx) \]. Why?
Every drink (in our fridge) is either a beer or a lemonade 
$(\forall x)(Dx \supset (Bx \lor Lx))$. So either every drink is a beer or some drink is a lemonade $(\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)$. Why? Take an arbitrary drink.
Every *drink* (in our fridge) is either a *beer* or a *lemonade* 
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Every \textit{drink} (in our fridge) is either a \textit{beer} or a \textit{lemonade} 

\((\forall x)(Dx \supset (Bx \lor Lx))\). So either every \textit{drink} is a \textit{beer} or some \textit{drink} is a \textit{lemonade} 

\((\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)\). Why? Take an arbitrary \textit{drink}. It’s either a \textit{beer} or a \textit{lemonade}. If it’s a \textit{lemonade}, we have the conclusion that some \textit{drink} is a \textit{lemonade}. 
Every drink (in our fridge) is either a beer or a lemonade
\((\forall x)(Dx \supset (Bx \lor Lx))\). So either every drink is a beer or some drink is a lemonade \((\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)\). Why? Take an arbitrary drink. It’s either a beer or a lemonade. If it’s a lemonade, we have the conclusion that some drink is a lemonade. If we don’t have that conclusion, then that arbitrary drink is a beer.
Every *drink* (in our fridge) is either a *beer* or a *lemonade* $(\forall x)(Dx \supset (Bx \lor Lx))$. So either every *drink* is a *beer* or some *drink* is a *lemonade* $(\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)$. Why? Take an arbitrary *drink*. It’s either a *beer* or a *lemonade*. If it’s a *lemonade*, we have the conclusion that some *drink* is a *lemonade*. If we don’t have that conclusion, then that arbitrary *drink* is a *beer*, and so, all the *drinks* are *beers*.
Every drink (in our fridge) is either a beer or a lemonade \((\forall x)(Dx \supset (Bx \lor Lx))\). So either every drink is a beer or some drink is a lemonade \((\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)\). Why? Take an arbitrary drink. It’s either a beer or a lemonade. If it’s a lemonade, we have the conclusion that some drink is a lemonade. If we don’t have that conclusion, then that arbitrary drink is a beer, and so, all the drinks are beers, and so, we also have our conclusion.
Example Proof (the formal structure)
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\[
\begin{align*}
&\quad \text{Ba} \supset \text{Ba} \quad \text{La} \supset \text{La} \\
&\quad \text{Da} \supset \text{Da} \quad \text{Ba} \lor \text{La} \supset \text{Ba}, \text{La} \\
&\quad \text{Da} \supset \text{Da} \quad \text{Da} \to (\text{Ba} \lor \text{La}), \text{Da} \supset \text{Ba}, \text{La} \\
&\quad \text{Da} \to (\text{Ba} \lor \text{La}), \text{Da} \supset \text{Ba}, \text{Da} \land \text{La} \\
&\quad \text{Da} \to (\text{Ba} \lor \text{La}) \supset \text{Da} \to \text{Ba}, \text{Da} \land \text{La} \\
&\quad (\forall x)(\text{D}x \to (\text{B}x \lor \text{L}x)) \supset \text{Da} \to \text{Ba}, (\exists x)(\text{D}x \land \text{L}x) \\
&\quad (\forall x)(\text{D}x \to (\text{B}x \lor \text{L}x)) \supset (\forall x)(\text{D}x \to \text{B}x), (\exists x)(\text{D}x \land \text{L}x) \\
&\quad (\forall x)(\text{D}x \to (\text{B}x \lor \text{L}x)) \supset (\forall x)(\text{D}x \to \text{B}x) \lor (\exists x)(\text{D}x \land \text{L}x)
\end{align*}
\]
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But what I say here can be extended to proof relying on other concepts.
Puzzles about proof

- How can proofs expand our knowledge, when the conclusion is \textit{(in some sense)} already present in the premises?
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- How can we be ignorant of a conclusion which logically follows from what we already know?
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- What grounds the necessity in the connection between the premises and the conclusion?
Puzzles about proof

- How can proofs expand our knowledge, when the conclusion is\textit{(in some sense)} already present in the premises?
- How can we be ignorant of a conclusion which logically follows from what we already know?
- What\textit{grounds} the necessity in the connection between the premises and the conclusion?
- (Notice that these are important questions for proofs in first order predicate logic, as much as for proof more generally.)
BACKGROUND
Assertions and Denials

[\[X : Y\]]
... in a communicative practice

Assertions and denials are moves in a practice.
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I can *deny* what you *assert*.
Assertions and denials are moves in a practice.

I can *deny* what you *assert*.

I can *retract* an assertion or a denial.
Assertions and denials are moves in a practice.

I can *deny* what you *assert*.

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I can ‘try on’ assertion or denial hypothetically.
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I can *deny* what you *assert*.

I can *retract* an assertion or a denial.

I can ‘try on’ assertion or denial hypothetically.

They are connected to other speech acts, too, like imperatives, interrogatives, recognitives, observatives, etc.
Assertions and denials take a *stand* (pro or con) on something.

DENIAL clashes with assertion.

ASSERTION clashes with denial.
These norms give rise to BOUNDS for positions.
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**IDENTITY:** $[A : A]$ is out of bounds.
These norms give rise to bounds for positions.

**Identity:** \([A : A]\) is out of bounds.

**Weakening:** If \([X : Y]\) is out of bounds, then \([X, A : Y]\) and \([X : A, Y]\) are also out of bounds.
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**CUT:** If \([X, A : Y]\) and \([X : A, Y]\) are out of bounds, then so is \([X : Y]\).
These norms give rise to BOUNDS for positions.

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**CUT:** If \([X, A : Y]\) and \([X : A, Y]\) are out of bounds, then so is \([X : Y]\).

A position that is OUT OF BOUNDS doesn’t succeed in taking a stand.
Explicit Definition

Define a concept by showing how you can compose that concept out of more primitive concepts.
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\[ x \text{ is a square} \quad =_{\text{df}} \quad x \text{ is a rectangle} \land \text{all sides of } x \text{ are equal in length.} \]
Explicit Definition

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Concepts defined explicitly are *sharply delimited* (contingent on the definition).
Explicit Definition

Define a concept by showing how you can compose that concept out of more primitive concepts.

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Concepts defined explicitly are *sharply delimited* (contingent on the definition).

Logical concepts are similarly sharply delimited, but they cannot all be given explicit definitions.
Definition through a rule for use

\[[X, A \land B : Y] \text{ is out of bounds}\]

if and only if

\[[X, A, B : Y] \text{ is out of bounds}\]
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\[ [X, A \land B : Y] \text{ is out of bounds} \]

if and only if

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\[
\frac{X, A, B \succ Y}{X, A \land B \succ Y} \quad \text{\textit{\textdegree Df}}
\]
What about when to deny a conjunction?

\[
\begin{align*}
A \land B &\vdash A \land B & \text{Id} \\
X \vdash B, Y &\quad A, B \vdash A \land B & \land \text{Df} \\
X \vdash A, Y &\quad X, A \vdash A \land B, Y & \text{Cut} \\
X \vdash A \land B, Y & & \text{Cut}
\end{align*}
\]
What about when to *deny* a conjunction?

\[
\begin{align*}
A \land B & \vdash A \land B & \text{Id} \\
X \vdash B, Y & A, B \vdash A \land B & \land \text{Df} \\
X \vdash A, Y & X, A \vdash A \land B, Y & \text{Cut} \\
\hline
X \vdash A \land B, Y & X \vdash A \land B, Y & \text{Cut} \\
\hline
\end{align*}
\]

So, we have

\[
\begin{align*}
X \vdash A, Y & X \vdash B, Y & \land \text{R} \\
\hline
X \vdash A \land B, Y & X \vdash A \land B, Y & \text{Cut} \\
\end{align*}
\]
Definitions for other logical concepts

\[
\begin{align*}
X \not\vdash A, Y & \quad \rightarrow \text{Df} \\
X, \neg A \not\vdash Y & \quad \rightarrow \text{Df} \\
X, A \not\vdash B, Y & \quad \rightarrow \text{Df} \\
X \not\vdash A \rightarrow B, Y & \quad \rightarrow \text{Df} \\
X \not\vdash A \lor B, Y & \quad \lor \text{Df}
\end{align*}
\]
Definitions for other logical concepts

\[
\begin{align*}
X \vdash A, Y & \quad \neg \text{Df} \\
X, \neg A & \vdash Y \\
\hline
X, A & \vdash B, Y \quad \rightarrow \text{Df} \\
X \vdash A \rightarrow B, Y \\
\hline
X \vdash A, B, Y \quad \lor \text{Df} \\
X \vdash A \lor B, Y
\end{align*}
\]

\[
\begin{align*}
X \vdash A \times n, Y & \quad \forall \text{Df} \\
X \vdash (\forall x) A, Y \\
\hline
X, A \times n & \vdash Y \quad \forall \text{Df} \\
X, (\exists x) A & \vdash Y \\
\hline
X, Fs & \vdash Ft, Y \quad = \text{Df} \\
X \vdash s = t, Y
\end{align*}
\]

(Where \( n \) and \( F \) are not present in \( X \) and \( Y \).)
Concepts defined in this way...

- Are *uniquely defined*. (If you and I use the same rule, we define the same concept.)
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- Are **conservatively extending**. (Adding a logical concept to your vocabulary in this way doesn’t constrain the bounds in the original language.)
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- Are *uniquely defined*. (If you and I use the same rule, we define the same concept.)

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- *Play useful dialogical roles*. (You can do things with these concepts that you cannot do without. Denying a conjunction does something different to denying the conjuncts.)

- Are *subject matter neutral*. (They work wherever you assert and deny—and have singular terms and predicates.)

In Brandom’s terms, they make explicit some of what was implicit in the practice of assertion and denial.
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- In Brandom’s terms, they *make explicit* some of what was implicit in the practice of assertion and denial.
WHAT PROOFS ARE & WHAT THEY DO
If it’s Thursday, I’m in Melbourne.

It’s Thursday.

Therefore, I’m in Melbourne.
If it’s Thursday, I’m in Melbourne.

It’s Thursday.

Therefore, I’m in Melbourne.

\[
\frac{A \rightarrow B, \ A \succ B}{A \rightarrow B, \ A \succ B} \quad \rightarrow \text{Def}
\]
If it’s Thursday, I’m in Melbourne.

It’s Thursday.

Therefore, I’m in Melbourne.

\[
\begin{align*}
A \to B &\implies A \to B \\
\overline{\quad} &\implies A \to B, A \implies B \\
\overline{\quad} &\implies A \to B, A \implies B
\end{align*}
\]

[It’s Thursday → I’m in Melbourne, It’s Thursday : I’m in Melbourne]

(This is out of bounds.)
The Undeniable

Take a context in which I’ve asserted it’s Thursday → I’m in Melbourne and I’ve asserted it’s Thursday, then I’m in Melbourne is undeniable.
Take a context in which I’ve asserted 
\[ \text{it’s Thursday} \rightarrow \text{I’m in Melbourne} \]
and I’ve asserted \text{it’s Thursday}, then \text{I’m in Melbourne is undeniable}.

Adding the \textit{assertion} makes explicit what was \textit{implicit} before that assertion.
The Undeniable

Take a context in which I’ve asserted

\textit{it’s Thursday} \rightarrow \textit{I’m in Melbourne}

and I’ve asserted \textit{it’s Thursday},
then \textit{I’m in Melbourne} is undeniable.

Adding the assertion makes explicit
what was \textit{implicit} before that assertion.

The \textit{stance} (pro or con)
on \textit{I’m in Melbourne} was already made.
A proof for $X \nrightarrow Y$ shows that the position $[X : Y]$ is out of bounds, by way of the defining rules for the concepts involved in the proof.
A proof for $X \rightarrow Y$ shows that the position $[X : Y]$ is out of bounds, by way of the defining rules for the concepts involved in the proof.

In this sense, proofs are *analytic*. They apply, given the definitions, independently of the positions taken by those giving the proof.
A proof of $A, B \rightarrow C, D$ can be seen as a proof of $C$ from $[A, B : D]$. 
A proof of $A, B \rightarrow C, D$ can be seen as a *proof* of $C$ from $[A, B : D]$, and a *refutation* of $A$ from $[B : C, D]$, and *more*. 
COUNTEREXAMPLES & KREISEL’S SQUEEZE
Enlarging Positions

\[\frac{X \rhd A, Y \quad X, A \rhd Y}{X \rhd Y} \quad \text{Cut}\]
Enlarging Positions

\[
\frac{X \not\rightarrow A, Y \quad X, A \not\rightarrow Y}{X \not\rightarrow Y} \quad \text{Cut}
\]

If \(X \not\rightarrow Y\) is not derivable then one of \(X, A \not\rightarrow Y\) and \(X \not\rightarrow A \not\rightarrow Y\) is also not derivable.
Enlarging Positions

\[
\frac{X \vdash A, Y \quad X, A \vdash Y}{X \vdash Y} \quad \text{Cut}
\]

If \( X \vdash Y \) is not derivable
then one of \( X, A \vdash Y \) and \( X \vdash A, Y \)
is also not derivable.

If \([X : Y]\) is available, then
so is either \([X, A : Y]\) or \([X : A, Y]\)
If \([X : Y]\) is available, we can extend it into a partition \([X' : Y']\) of the entire language.
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\(U \rightarrow V\) is not derivable for any finite \(U \subseteq X'\) and \(V \subseteq Y'\).
Adding Witnesses

If $(\exists x)A$ is added on the left, we also add a witness $A|_n^x$, where $n$ is fresh.
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If $(\exists x)A$ is added on the left, we also add a witness $A|_n^x$, where $n$ is fresh.

\[
\frac{X, A|_n^x, (\exists x)A \rightarrow Y}{X, (\exists x)A \rightarrow Y} \quad \exists DfW
\]
Adding Witnesses

If $(\exists x)A$ is added on the left, we also add a witness $A|_{n}^{x}$, where $n$ is fresh and similarly when $(\forall x)A$ is added on the right.

$$\frac{X, A|_{n}^{x}, (\exists x)A \succ Y}{X, (\exists x)A \succ Y} \quad \exists Df, W$$

$$\frac{X \succ (\forall x)A, A|_{n}^{x}, Y}{X \succ (\forall x)A, Y} \quad \forall Df, W$$
Witnessed Limit Positions give rise to Models

A \not\in X' \iff A \notin X' \\
A \in B \in X' \iff A \in X' \land B \in X' \\
A \in \notin B \in X' \iff A \in X' \lor B \in X' \\
A \not\in B \in X' \iff A \in Y' \lor B \in X' \\

(8x) A \in X' \iff A \in X' \land \text{for each name } n. \\
(9x) A \in X' \iff A \in X' \land \text{for some name } n. \\

This is a model, where the true formulas are in X' and the false formulas are in Y', and whose domain is the set of names. (Things are little more delicate when the language contains the identity predicate.)
Witnessed Limit Positions give rise to Models

\[ A \in X' \iff \neg A \notin X' \iff \neg A \in Y' , \]
\[ A \land B \in X' \iff A \in X' \text{ and } B \in X' . \]
\[ A \lor B \in X' \iff A \in X' \text{ or } B \in X' . \]
\[ A \rightarrow B \in X' \iff A \in Y' \text{ or } B \in X' . \]
\[ (\forall x)A \in X' \iff A|^{x}_{n} \in X' \text{ for each name } n . \]
\[ (\exists x)A \in X' \iff A|^{x}_{n} \in X' \text{ for some name } n . \]
Witnessed Limit Positions give rise to Models

\[
A \in X' \text{ iff } \neg A \notin X' \text{ iff } \neg A \in Y',
\]

\[
A \land B \in X' \text{ iff } A \in X' \text{ and } B \in X'.
\]

\[
A \lor B \in X' \text{ iff } A \in X' \text{ or } B \in X'.
\]

\[
A \rightarrow B \in X' \text{ iff } A \in Y' \text{ or } B \in X'.
\]

\[
(\forall x)A \in X' \text{ iff } A|_n^x \in X' \text{ for each name } n.
\]

\[
(\exists x)A \in X' \text{ iff } A|_n^x \in X' \text{ for some name } n.
\]

This is a model, where the true formulas are in \(X'\) and the false formulas are in \(Y'\), and whose domain is the set of names.
Witnessed Limit Positions give rise to Models

\[ A \in X' \text{ iff } \neg A \notin X' \text{ iff } \neg A \in Y', \]
\[ A \land B \in X' \text{ iff } A \in X' \text{ and } B \in X'. \]
\[ A \lor B \in X' \text{ iff } A \in X' \text{ or } B \in X'. \]
\[ A \rightarrow B \in X' \text{ iff } A \in Y' \text{ or } B \in X'. \]
\[ (\forall x)A \in X' \text{ iff } A|_n^x \in X' \text{ for each name } n. \]
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This is a model, where the true formulas are in \( X' \) and the false formulas are in \( Y' \), and whose domain is the set of names.

(Things are little more delicate when the language contains the identity predicate.)
X ⊬ Y is derivable
iff there is no *model*
in which each member of X is true
and each member of Y is false.
X ▷ Y is informally valid
Kreisel's *Squeeze*

\[ X \supset Y \text{ has a derivation} \]

\[ \Downarrow \]

\[ X \supset Y \text{ is *informally* valid} \]
Kreisel's *Squeeze*

\[
X \rightarrow Y \text{ has a derivation} \\
\Downarrow \\
X \rightarrow Y \text{ is } \textit{informally} \text{ valid} \\
\Downarrow \\
X \rightarrow Y \text{ has no countermodel}
\]
Kreisel's *Squeeze*

\[ X \vdash Y \text{ has a derivation} \]

\[ \Downarrow \]

\[ X \vdash Y \text{ is } \text{informally} \text{ valid} \]

\[ \Downarrow \]

\[ X \vdash Y \text{ has no countermodel} \]

\[ \Downarrow \]

\[ X \vdash Y \text{ has a derivation.} \]
To say that $X \vdash Y$ is informally valid means that is a clash involved in asserting each member of $X$ and denying each member of $Y$. 

Axiomatic sequents are informally valid in this sense. Structural rules preserve informal validity. Defining rules define the connectives/quantifiers.
To say that $X \Rightarrow Y$ is *informally valid* means that is a clash involved in asserting each member of $X$ and denying each member of $Y$.

Axiomatic sequents ($A \Rightarrow A$) are informally valid in this sense.
To say that $X \to Y$ is *informally valid* means that is a clash involved in asserting each member of $X$ and denying each member of $Y$.

Axiomatic sequents $(A \to A)$ are informally valid in this sense.

Structural rules preserve informal validity.
To say that $X \rightarrow Y$ is *informally valid* means that is a clash involved in asserting each member of $X$ and denying each member of $Y$.

- Axiomatic sequents $(A \rightarrow A)$ are informally valid in this sense.

- Structural rules preserve informal validity.

- Defining rules *define* the connectives/quantifiers.
Refine our notion of informal validity: Literals (\(Fa\), \(Gb\c\), etc.) are informally logically independent. We ignore logical connections between literals—we fix on informal validity in virtue of first order logical form.

Given a witnessed partition position \([X:Y]\) (i.e., given a model), there is no informal clash (in virtue of logical form) involved in asserting any of the literals in \(X\) and denying any in \(Y\).

So, there is no clash involved in asserting any formulas in \(X\) and denying any formulas in \(Y\), by appeal to the defining rules. (This is an induction on the depth of the structure of the formulas. The defining rules reduce clashes involving formulas into clashes involving subformulas.)

So, a countermodel for a sequent shows how there is no clash involved in asserting each member of \(X\) and denying each member of \(Y\).
(2) From Countermodel to Informal Invalidity

Refine our notion of informal validity: Literals (Fa, Gbc, etc.) are informally logically independent. We ignore logical connections between literals—we fix on informal validity in virtue of first order logical form.

Given a witnessed partition position \([X : Y]\) (i.e., given a model), there is no informal clash (in virtue of logical form) involved in asserting any of the literals in \(X\) and denying any in \(Y\).

So, there is no clash involved in asserting any formulas in \(X\) and denying any formulas in \(Y\), by appeal to the defining rules. (This is an induction on the depth of the structure of the formulas. The defining rules reduce clashes involving formulas into clashes involving subformulas.)

So, a countermodel for a sequent shows how there is no clash involved in asserting each member of \(X\) and denying each member of \(Y\).
Refine our notion of informal validity: *Literals* (Fa, Gbc, etc.) are informally logically independent. We *ignore* logical connections between literals—we fix on informal validity *in virtue of first order logical form*. So, there is no clash involved in asserting any formulas in \( X \) and denying any formulas in \( Y \). By appeal to the defining rules, this is an induction on the depth of the structure of the formulas. The defining rules reduce clashes involving formulas into clashes involving subformulas. A countermodel for a sequent shows how there is no clash involved in asserting each member of \( X \) and denying each member of \( Y \).
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Given a witnessed partition position \([X : Y]\) (i.e., given a model), there is no informal clash (in virtue of logical form) involved in asserting any of the literals in \(X\) and denying any in \(Y\).
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So, a countermodel for a sequent shows *how* there is no clash involved in asserting each member of X and denying each member of Y.
That’s the *Completeness Theorem*. 
Kreisel's Squeeze

$X \Rightarrow Y$ has a derivation

\[\Downarrow\]

$X \Rightarrow Y$ is informally valid

\[\Downarrow\]

$X \Rightarrow Y$ has no countermodel

\[\Downarrow\]

$X \Rightarrow Y$ has a derivation.
Informal validity (in virtue of first order logical form), for the language given by the defining rules, is first order classical logic, as given by the sequent calculus and Tarski’s models.
CONSEQUENCES FOR HOW PROOFS WORK
Observation 0: Proofs are *definitionally analytic*

Validity is grounded in the *rules* *defining* the concepts used in them.
Observation 1: Proofs Preserve Truth

- The account of consequence does not use the concept of *truth*. 

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However, given plausible (minimal) assumptions concerning $T$, we can show that *(for example)* if $A, B \vdash C$ then $T\langle A \rangle, T\langle B \rangle \vdash T\langle C \rangle$. 
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- However, given plausible (minimal) assumptions concerning $T$, we can show that (for example) if $A, B \rightarrow C$ then $T\langle A \rangle, T\langle B \rangle \rightarrow T\langle C \rangle$.

- This *follows from* the concepts of consequence and truth.
Observation 2: *Specification* outstrips *Recognition*

Our ability to *specify* concepts and consequence far outstrips our ability to *recognise* that consequence.
SUCCESSOR AXIOMS:
PA1: $\forall x \forall y (x' = y' \rightarrow x = y)$;
PA2: $\forall x (0 \neq x')$.

ADDITION AXIOMS:
PA3: $\forall x (x + 0 = x)$;
PA4: $\forall x (x + y' = (x + y)')$.

MULTIPLICATION AXIOMS:
PA5: $\forall x (x \times 0 = 0)$;
PA6: $\forall x \forall y (x \times y' = (x \times y) + x)$.

INDUCTION SCHEME:
PA7: $(\phi(0) \land \forall x (\phi(x) \rightarrow \phi(x'))) \rightarrow \forall x \phi(x)$.

GOLDBACH’S CONJECTURE:
GC: $\forall x \exists y \exists z (\text{Prime } y \land \text{Prime } z \land 0'' \times x = y + z)$
Observation 2: *Specification* outstrips *Recognition*

Is \([\text{PA1}, \ldots, \text{PA7} : \text{GC}]\) out of bounds?
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Is \([\text{PA}1, \ldots, \text{PA}7 : \text{GC}]\) out of bounds?

We have no idea.
Observation 2: Specification outstrips Recognition

Is \([p_{A1}, \ldots, p_{A7} : GC]\) out of bounds?

We have no idea.

This is not a \textit{bug}. It’s a \textit{feature}. 
Observation 2: *Specification* outstrips *Recognition*

Is \([\text{PA1}, \ldots, \text{PA7} : \text{GC}]\) out of bounds?

We have no idea.

This is not a *bug*. It’s a *feature*.

Our concepts are rich and expressive.
We can say things whose significance we continue to work out.
Observation 2: *Specification* outstrips *Recognition*

Is $[\text{PA}_1, \ldots, \text{PA}_7 : \text{GC}]$ out of bounds?

We have no idea.

This is not a *bug*. It’s a *feature*.

Our concepts are rich and expressive.
We can say things whose significance we continue to work out.

Verifying a putative proof is straightforward.
Checking that something *has* a proof is not so easy.
Suppose that $\text{PA} \vdash \text{GC}$ is derivable (but we don’t possess that proof) and that we *know* $\text{PA}$.

Do we know $\text{GC}$?
In a weak sense of ‘know’, yes, we do know GC

- It’s a logical consequence of what we know.
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- It’s a logical consequence of what we know.
- It is implicitly present in what we already know.
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- There is no epistemic possibility (no circumstance consistent with our knowledge) that leaves GC out.
In a weak sense of ‘know’, yes, we do know GC

- It’s a logical consequence of what we know.
- It is implicitly present in what we already know.
- There is no epistemic possibility (no circumstance consistent with our knowledge) that leaves GC out.
- The means to come to know GC (the derivation) is “there” to be found.
In a not-so-weak sense, we don't know GC

- Do we believe GC?
In a not-so-weak sense, we don't know GC

- Do we believe GC?
- If we do believe it, do we believe it in the right way?
In a not-so-weak sense, we don't know GC

- Do we believe GC?
- If we do believe it, do we believe it *in the right way*?
- There “is” evidence for GC (its proof from PA, for example), but if that evidence plays no role in our belief...
Observation 3: Proofs Can Transfer Warrant

- The account of consequence does not use the concept of *warrant*.
Observation 3: Proofs Can Transfer Warrant

- The account of consequence does not use the concept of warrant.

- However, given plausible (less minimal) assumptions concerning warrant, we can show that (for example) if \( p \) is a proof for \( A, B \vdash C \) then \( x : A, y : B \vdash p(x, y) : C \).
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- Here, \( p \) transforms warrants for the premises into warrant for the conclusion.
Observation 3: Proofs Can Transfer Warrant

- The account of consequence does not use the concept of warrant.

- However, given plausible (less minimal) assumptions concerning warrant, we can show that (for example) if \( p \) is a proof for \( A, B \models C \) then \( x : A, y : B \models p(x, y) : C \).

- Here, \( p \) transforms warrants for the premises into warrant for the conclusion.

- This works only for categorical, conclusive warrants (grounds), not for defeasible warrants.
A Caveat on Defeasible Warrants

Consider the “Lottery Paradox.”
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\[
\begin{align*}
(\exists x)(Tx \wedge Wx), \\
(\forall x)(Tx &\equiv (x = t_1 \lor x = t_2 \lor \cdots \lor x = t_{1,000,000})) \\
: \quad Wt_1, Wt_2, \ldots, Wt_{1,000,000}
\end{align*}
\]
Consider the “Lottery Paradox.”

\[
\begin{align*}
&\exists x (T_x \land W_x), \\
&\forall x (T_x \equiv (x = t_1 \lor x = t_2 \lor \cdots \lor x = t_{1,000,000})) \\
&: W_{t_1}, W_{t_2}, \ldots, W_{t_{1,000,000}}
\end{align*}
\]

We have a very high degree of confidence in each part.

Each component is highly probable.

But the whole position is out of bounds.
“Well, now, let’s take a little bit of the argument in that First Proposition—just two steps, and the conclusion drawn from them. Kindly enter them in your note-book. And in order to refer to them conveniently, let’s call them $A$, $B$, and $Z:\Rightarrow$

$(A)$ Things that are equal to the same are equal to each other.

$(B)$ The two sides of this Triangle are things that are equal to the same.

$(Z)$ The two sides of this Triangle are equal to each other.

Readers of Euclid will grant, I suppose, that $Z$ follows logically from $A$ and $B$, so that any one who accepts $A$ and $B$ as true, must accept $Z$ as true?”

“Undoubtedly! The youngest child in a High School—as soon as High Schools are invented, which will not be till some two thousand years later—will grant that.”

“And if some reader had not yet accepted $A$ and $B$ as true, he might still accept the sequence as a valid one, I suppose?”
"No doubt such a reader might exist. He might say ‘I accept as true the Hypothetical Proposition that, if \( A \) and \( B \) be true, \( Z \) must be true; but, I don’t accept \( A \) and \( B \) as true.’ Such a reader would do wisely in abandoning Euclid, and taking to football.”

“And might there not also be some reader who would say ‘I accept \( A \) and \( B \) as true, but I don’t accept the Hypothetical’?”

“Certainly there might. \( He, \) also, had better take to football.”

“And neither of these readers,” the Tortoise continued, “is as yet under any logical necessity to accept \( Z \) as true?”

“Quite so,” Achilles assented.

“Well, now, I want you to consider me as a reader of the second kind, and to force me, logically, to accept \( Z \) as true.”

“A tortoise playing football would be——” Achilles was beginning

“—an anomaly, of course,” the Tortoise hastily interrupted. “Don’t wander from the point. Let’s have \( Z \) first, and football afterwards!”

“I’m to force you to accept \( Z \), am I?” Achilles said musingly. “And your present position is that you accept \( A \) and \( B \), but you don’t accept the Hypothetical——”

“Let’s call it \( C \),” said the Tortoise.

“—but you don’t accept

\[
(C) \quad \text{If } A \text{ and } B \text{ are true, } Z \text{ must be true.}
\]

“That is my present position,” said the Tortoise.

“Then I must ask you to accept \( C \).”
This doesn't mean when I accept \( A \) and I accept \( A \to Z \), I ought to also accept \( Z \).

However, if I assert \( A \) and \( A \to Z \) then \( Z \) is undeniable.
This doesn’t mean when I accept $A$ and I accept $A \rightarrow Z$, I ought to also accept $Z$. 

$A, A \rightarrow Z \not\rightarrow Z$
A, A → Z ⊨ Z

This *doesn’t* mean when I accept A and I accept A → Z, I ought to also accept Z.

However, if I assert A and A → Z then Z is *undeniable*.
If I assert $A$ and $if \ A \ then \ Z$ and deny $Z$, then I am using ‘if ...then’ in a way that deviates from the defining rule for $\rightarrow$, or I am explicitly contradicting myself.
If I assert $A$ and *if* $A$ *then* $Z$ and *deny* $Z$, then I am using ‘*if ... then*’ in a way that deviates from the defining rule for $\rightarrow$, or I am explicitly contradicting myself.

\[
\begin{align*}
A \rightarrow B & \succ A \rightarrow B \\
\frac{A \rightarrow B, A \succ B}{\rightarrow Df}
\end{align*}
\]
If I assert \( A \) and \( if \ A \ then \ Z \) and ask whether \( Z \) holds?

- We need to understand connections between defining rules and norms for questions and answers, as well as assertions and denials.
An account of the logical concepts given in terms of defining rules governing assertions and denials helps explain how (first order predicate logic) proof works, how possessing a proof can expand our knowledge, while proofs make explicit what is implicit in what we know.
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An account of the logical concepts given in terms of defining rules governing assertions and denials helps explain how (first order predicate logic) proof works, how possessing a proof can expand our knowledge, while proofs make explicit what is implicit in what we know.
THANK YOU!

https://consequently.org/presentation/2018/what-proofs-are-for-nyu/

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